Decomposition-based Multiobjective Optimization for Constrained Evolutionary Optimization

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***Abstract*—Pareto dominance-based multiobjective optimization has been successfully applied to constrained evolutionary optimization during the last two decades. However, as another famous multiobjective optimization framework, decomposition-based multiobjective optimization has not received sufficient attention from constrained evolutionary optimization. In this paper, we make use of decomposition-based multiobjective optimization to solve constrained optimization problems. In our method, first of all, a constrained optimization problem is transformed into a biobjective optimization problem. Afterward, the transformed biobjective optimization problem is decomposed into a number of scalar optimization subproblems. After generating an offspring for each subproblem by differential evolution, the weighted sum method is utilized for selection. In addition, to make decomposition-based multiobjective optimization suit the characteristics of constrained evolutionary optimization, weight vectors are elaborately adjusted. Moreover, for some extremely complicated constrained optimization problems, a restart strategy is introduced to help the population jump out of a local optimum in the infeasible region. Extensive experiments on three sets of benchmark test functions, namely, 24 test functions from IEEE CEC2006, 36 test functions from IEEE CEC2010, and 56 test functions from IEEE CEC2017, have demonstrated that the proposed method shows better or at least competitive performance against other state-of-the-art methods.**

***Index Terms*—constrained optimization problems, multiobjective optimization, decomposition, Pareto dominance, evolutionary algorithms**

I. Introduction

MANY scientific and engineering optimization problems can be formulated as constrained optimization problems (COPs) [1], [2]. Without loss of generality, a COP can be described as:

$$\begin{aligned} &\text{minimize}\, f(\vec{x}),\ \vec{x}=(x_1,\ldots,x_D)\in S,\ L_i\le x_i\le U_i\\ &\text{subject to}: g_j(\vec{x})\le 0,\ j=1,\ldots,l\\ &\qquad\qquad\quad h_j(\vec{x})=0,\ j=l+1,\ldots,m \end{aligned}$$

where $f(\vec{x})$ is the objective function, $\vec{x}$ is the decision vector (a solution or an individual), x_i is the ith dimension of $\vec{x}$, D is the number of dimensions, L_i and U_i are the lower and upper bounds of x_i, respectively, $S=\prod_{i=1}^{D}[L_i,U_i]$ is the decision space, $g_j(\vec{x})$ is the jth inequality constraint, l is the number of inequality constraints, $h_j(\vec{x})$ is the $(j-l)$th equality constraint, and $(m-l)$ is the number of equality constraints.

For COPs, the degree of constraint violation of $\vec{x}$ on the jth constraint is expressed as follows:

$$G_j(\vec{x})=\begin{cases}\max(0,g_j(\vec{x})), & 1\le j\le l\\ \max(0,|h_j(\vec{x})|-\delta), & l+1\le j\le m\end{cases} \tag{1}$$

where δ is a positive tolerance value to relax equality constraints. Afterward, the degree of constraint violation of $\vec{x}$ on all constraints is calculated as follows:

$$G(\vec{x})=\sum_{j=1}^{m}G_j(\vec{x}) \tag{2}$$

$\vec{x}$ is called a feasible solution if $G(\vec{x})=0$; otherwise, it is called an infeasible solution. The goal of solving a COP is to locate the feasible optimum.

In the community of evolutionary computation, there has been an increasing interest in applying evolutionary algorithms (EAs) to solve COPs. In order for EAs to deal with COPs, constraint-handling techniques should be integrated. In principle, EAs aim to generate offspring while constraint-handling techniques are in charge of comparing individuals. The last two decades have witnessed the successful applications of multiobjective optimization to design constraint-handling techniques. In multiobjective optimization-based constraint-handling techniques, a COP is first transformed into a multiobjective optimization problem (MOP). Then, multiobjective optimization techniques are used to compare individuals. In this paper, multiobjective optimization-based constraint-handling techniques are briefly classified into three categories: 1) standard multiobjective optimization methods, 2) standard biobjective optimization methods, and 3) generalized multiobjective optimization methods. The standard multiobjective optimization methods transform a COP into a MOP with $(m+1)$ objectives, i.e., $(f(\vec{x}),G_1(\vec{x}),\ldots,G_m(\vec{x}))$. Multiobjective optimization-based constraint-handling techniques at the early stage always fall into this category [3], [4]. Under

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this condition, the transformed MOP always involves more than two objectives. As we know, MOPs with more than two objectives usually exhibit complicated properties. Consequently, the transformed MOP is also very difficult to be tackled as the original COP. In contrast, the standard biobjective optimization methods consider the degree of constraint violation, i.e., $G(\vec{x})$, as another objective function in addition to the original objective function $f(\vec{x})$. Interestingly, most of the recent multiobjective optimization-based constraint-handling techniques belong to this category [5]–[9]. Different from the above two categories, the generalized multiobjective optimization methods introduce other additional objective functions or constraints [10], [11].

It can be found that most of the existing multiobjective optimization-based constraint-handling techniques are based on Pareto dominance [12], in which Pareto dominance is viewed as the criterion to compare individuals. Note, however, that decomposition is another famous multiobjective optimization framework [13]–[15]. Its outperformed performance has been demonstrated in a lot of literature such as [16], [17], and [18]. Different from Pareto dominance-based multiobjective optimization, decomposition-based multiobjective optimization decomposes a MOP into a set of scalar optimization subproblems, where each subproblem is assigned a weight vector. Afterward, these subproblems are optimized in a collaborative manner. Such a framework exhibits numerous advantages for solving MOPs. By decomposing a MOP into a set of scalar optimization subproblems, every two solutions are comparable. Hence, a certain degree of selection pressure can be guaranteed. Besides, it is well-known that decomposition-based framework is more efficient than nondominated sorting [13], [19]. Moreover, by adjusting the weight vectors, search biases can be incorporated. Note that these search biases are crucial when taking advantage of multiobjective optimization to tackle COPs [20], [21]. However, little effort has been devoted to making use of the above advantages of decomposition-based multiobjective optimization for constrained evolutionary optimization.

Motivated by the above considerations, this paper makes an attempt to tailor decomposition-based multiobjective optimization to solve COPs. In our method, a COP is first converted into a biobjective optimization problem (BOP) ($f(\vec{x})$,$G(\vec{x})$). Afterward, this BOP is decomposed into NP scalar optimization subproblems. Each individual in the population is associated with a subproblem and is evolved along the direction defined by the weight vector of this subproblem. After generating an offspring for each subproblem by differential evolution (DE), the weighted sum method is employed to compare individuals. To make decomposition-based multiobjective optimization suit the properties of COPs, a weight vector adjusting strategy is designed. Furthermore, a restart strategy is introduced to cope with extremely complicated constraints. By the above process, an alternative constrained optimization EA (COEA), i.e., DeCODE, is proposed. Note that DeCODE is different from the constrained decomposition-based multiobjective optimization algorithm introduced in [22]. The algorithm in [22] utilizes constrained optimization to improve the decomposition-based method for multiobjective optimization. On the contrary, DeCODE applies the decomposition-based method to solve COPs.

The main contributions of this paper are summarized as follows:

- The idea of decomposition-based multiobjective optimization is thoroughly investigated for constrained evolutionary optimization.
- A weight vector adjusting strategy is designed to make the decomposition-based multiobjective optimization suit the properties of COPs.
- We develop a search algorithm to strike a balance not only between diversity and convergence, but also between constraints and objective function.
- A restart strategy is introduced to find feasible solutions for some COPs with extremely complicated constraints.

To be specific, in the theoretical aspect, the relationship between decomposition-based multiobjective optimization and constrained evolutionary optimization is analyzed. Besides, the weight vectors which are beneficial to solve COPs are clarified. Furthermore, how to achieve the tradeoff between constraints and objective function in a search algorithm is illustrated. In the practical aspect, systematic experiments on three benchmark test suites from IEEE CEC2006, IEE CEC2010, and IEEE CEC2017 have demonstrated that DeCODE is effective and efficient for solving various kinds of COPs.

The rest of this paper is organized as follows. Some preliminary knowledge is introduced in Section II. Section III conducts a brief survey of utilizing multiobjective optimization for constrained evolutionary optimization. The details of the proposed DeCODE are given in Section IV. Section V provides the empirical study. Finally, Section VI concludes this paper.

II. Preliminary Knowledge

Since decomposition-based multiobjective optimization is applied for constrained evolutionary optimization in this paper, some basic concepts of multiobjective optimization and the framework of decomposition-based multiobjective optimization are briefly introduced in this section.

A. Related Concepts of Multiobjective Optimization

In general, a MOP is formulated as follows:

$$\text{minimize}\,\vec{f}(\vec{x}) = (f_1(\vec{x}), \ldots, f_n(\vec{x})) \tag{3}$$

where $\vec{x} = (x_1, \ldots, x_D) \in S$ is a D-dimensional decision vector and $\vec{f}(\vec{x})$ is the objective vector involving n objective functions. Several related concepts of multiobjective optimization are presented below.

1) **Pareto Dominance:** A decision vector $\vec{x} = (x_1, \ldots, x_D)$ is said to Pareto dominate another decision vector $\vec{y} = (y_1, \ldots, y_D)$, denoted as $\vec{x} \prec \vec{y}$, if $\forall i \in \{1, \ldots, n\}$, $f_i(\vec{x}) \leq f_i(\vec{y})$ and $\vec{f}(\vec{x}) \neq \vec{f}(\vec{y})$.

2) **Pareto Optimum:** A decision vector $\vec{x}^*$ is called a Pareto optimal solution, if it is not Pareto dominated by any other decision vectors.

3) **Pareto Set:** The Pareto set PS is a set of all Pareto optimal solutions.

4) **Pareto Front:** The Pareto front is the image of the Pareto set in the objective space, i.e., $PF = \{\vec{f}(\vec{x}) | \vec{x} \in PS\}$.

The principal task of multiobjective optimization is to seek a reasonable approximation of the Pareto set/front.

B. Framework of Decomposition-based Multiobjective Optimization

Decomposition-based multiobjective optimization is very popular for solving MOPs [13]. Its framework with the weighted sum method [23] is described as follows [19]:

Step 1) Initialization:

Step 1.1) Set the archive which is used to store nondominated solutions as an empty set: $EP = \emptyset$.

Step 1.2) Initialize a uniform spread of NP weight vectors: $WV = \{\vec{\lambda}_1, \ldots, \vec{\lambda}_{NP}\}$, where $\vec{\lambda}_i = (\lambda_{i,1}, \ldots, \lambda_{i,n})$, $i \in \{1, \ldots, NP\}$.

Step 1.3) Calculate the mating neighborhood $B_m(i)$ which includes T_m indexes and the replacement neighborhood $B_r(i)$ which includes T_r indexes for each weight vector $\vec{\lambda}_i$ ($i \in \{1, \ldots, NP\}$).

Step 1.4) Generate a random population consisting of NP individuals: $P = \{\vec{x}_1, \ldots, \vec{x}_{NP}\}$, and evaluate the population: $FV = \{\vec{f}(\vec{x}_1), \ldots, \vec{f}(\vec{x}_{NP})\}$.

Step 1.5) Calculate the weighted sum of the population: $WS = \{g^{ws}(\vec{x}_1|\vec{\lambda}_1), \ldots, g^{ws}(\vec{x}_{NP}|\vec{\lambda}_{NP})\}$, where

$$g^{ws}(\vec{x}_i|\vec{\lambda}_i) = \sum_{j=1}^{n} \lambda_{i,j} f_j(\vec{x}_i), i \in \{1, \ldots, NP\} \tag{4}$$

Step 2) Population updating:

For $i = 1, \ldots, NP$ do

Step 2.1) Generate an offspring $\vec{y}$ for $\vec{x}_i$ by executing the search algorithm on several individuals selected based on $B_m(i)$.

Step 2.2) Evaluate $\vec{y}$: $\vec{f}(\vec{y}) = \{f_1(\vec{y}), \ldots, f_n(\vec{y})\}$.

Step 2.3) For each index $j \in B_r(i)$, if $g^{ws}(\vec{y}|\vec{\lambda}_j) \leq g^{ws}(\vec{x}_j|\vec{\lambda}_j)$, set $\vec{x}_j = \vec{y}$, $\vec{f}(\vec{x}_j) = \vec{f}(\vec{y})$, and $g^{ws}(\vec{x}_j|\vec{\lambda}_j) = g^{ws}(\vec{y}|\vec{\lambda}_j)$.

Step 2.4) Remove all the solutions Pareto dominated by $\vec{y}$ from EP, and add $\vec{y}$ into EP if no solutions in EP Pareto dominate $\vec{y}$.

Step 3) Stopping criterion: If the stopping criterion is satisfied, then stop and output EP; otherwise, go to **Step 2)**.

The above procedure explicitly decomposes a MOP into NP scalar optimization subproblems via the weighted sum method, as shown in (4). Each subproblem is associated with an individual and is optimized by making use of the information from its neighboring subproblems. The main idea behind decomposition-based multiobjective optimization is that the optimal solutions of neighboring subproblems should be close to each other and any information from one subproblem should be helpful for optimizing another subproblem.

III. Previous Work

A considerable number of multiobjective optimization-based constraint-handling techniques have been proposed during the last two decades. As mentioned previously, they are divided into three kinds in this paper: 1) standard multiobjective optimization methods, 2) standard biobjective optimization methods, and 3) generalized multiobjective optimization methods.

1) Standard multiobjective optimization methods: This kind of methods aims at optimizing $(f(\vec{x}), G_1(\vec{x}), \ldots, G_m(\vec{x}))$ simultaneously. Coello Coello *et al.* [24], [25] carried out a series of pioneer work on generalizing the classical multiobjective optimization EAs [26], [27] to solve COPs. Ray *et al.* [3], [28] calculated three ranks, which include the rank of objective function, the Pareto rank of constraints, and the Pareto rank of the combination of objective function and constraints. These three ranks are utilized to select solutions in a collaborative way. Angantyr *et al.* [29] proposed a constraint-handling technique which is a variant of a multiobjective real-coded genetic algorithm. In this method, the rank of objective function and the Pareto rank of constraints are calculated separately. Subsequently, these two ranks are aggregated together by the feasible proportion, i.e., the percentage of feasible solutions in the population. Aguirre *et al.* [4] modified the famous Pareto archived evolutionary strategy [30] to deal with COPs. In this method, the constrained search space is shrunk dynamically to focus the search effort on specific areas of the feasible region. Besides, an adaptive grid is utilized to store solutions.

2) Standard biobjective optimization methods: The aim of this kind of methods is to optimize the BOP $(f(\vec{x}), G(\vec{x}))$. Zhou *et al.* [31] defined the individual's Pareto strength, which is based on Pareto dominance. Afterward, a new real-coded genetic algorithm based on Pareto strength and minimal generation gap model is devised. In 2006, Cai and Wang [5] made use of Pareto dominance to compare individuals. Moreover, an infeasible solution archiving and replacement mechanism is proposed to drive the population approaching or landing in the feasible region quickly. Later, they improved this infeasible solution archiving and replacement mechanism based on multiobjective optimization and proposed CMODE [7]. In 2007, Wang *et al.* [6] proposed a hybrid COEA, called HCOEA, which effectively combines Pareto dominance with global and local search models. In [8], HCOEA is improved by dynamically implementing the global and local search models. In 2008, Wang *et al.* [32] divided the constrained optimization process into three phases. In the first phase, a selection strategy is designed based on Pareto dominance. Subsequently, several COEAs adopt or improve this three-phase-based method [33]–[36]. Similarly, Venkatraman and Yen [37] proposed a two-phase-based method to tackle COPs. In phase one, a COP is considered as a constraint satisfaction problem. In phase two, the famous nondominated sorting genetic algorithm II (NSGA-II) [38] and a niching scheme are combined to calculate the fitness value. Masuda and Kurihara [39] exploited the multiobjective optimization particle swarm optimization to solve COPs. In this method, only several Pareto optimal solutions with the least degree of constraint violation will be preserved if the number of Pareto optimal solutions exceeds a predefined threshold. In addition, a novel global best selection technique and a diversity preservation strategy are proposed. Deb and Datta [40] applied NSGA-II to estimate the penalty factor.

This study theoretically analyzes the relationship between the lower bound of the penalty factor and the slope of the Pareto front at the point of $G(\vec{x}) = 0$. Based on such analysis, the penalty factor is obtained. This method is further improved by estimating the penalty factor of each constraint separately [41]–[43]. Jiao *et al.* [9] proposed a novel selection strategy based on multiobjective optimization. In this method, Pareto dominance is used to classify dominated and nondominated solutions. Li and Zhang [21] pointed out that Pareto dominance lacks search biases toward constraints, which may lead to the inferior performance of a COEA. Afterward, the b-dominance is presented by introducing search biases into the conventional Pareto dominance.

3) Generalized multiobjective optimization methods: Watanabe and Sakakibara [44] presented two methods to transform a COP into a MOP: the first one considers a penalty function as an additional objective function and the second one adds noise to the original objective function or decision variables. Dong and Wang [10] converted a COP into the following BOP: $(f(\vec{x})+\varepsilon G(\vec{x}), G(\vec{x}))$. The theoretical analysis reveals that when ε tends to infinity, this BOP has the unique Pareto optimal vector, which exactly corresponds to the optimal solution of a COP. They claimed that this BOP could be solved by a traditional multiobjective optimization EA without biases. In the implementation phase, ε exponentially increases and Pareto ranking is employed as the selection criterion. Xu *et al.* [11] proposed a novel multiobjective model with helper objective functions for constrained optimization. In addition to $(f(\vec{x}), G(\vec{x}))$, an auxiliary objective function is constructed. And then a three-objective-based CMODE [7] is implemented. The experimental results show that the helper objective function is able to improve the performance of CMODE. Gao *et al.* [45] recast a COP as $(G_1(\vec{x}), \ldots, G_m(\vec{x}))$ with one constraint. In this method, the original objective function value is restricted to be less than a value which is set adaptively. Based on this formulation, a novel pair-wise comparison strategy is proposed. Li *et al.* [46] reformulated a COP as $(f(\vec{x}), G_1(\vec{x}), \ldots, G_m(\vec{x}))$ with dynamic constraints. Note that the original constraints are still kept into consideration in this method. To construct the dynamic environment, all constraints are bounded by a value which decreases with the increase of generation. Very recently, a general framework based on this idea is proposed to solve COPs in [47].

All the above-mentioned multiobjective optimization-based constraint-handling techniques are based on Pareto dominance due to the fact that Pareto dominance serves as the comparison criterion. As another generic multiobjective optimization framework, decomposition-based multiobjective optimization has been gaining increasing attention for solving MOPs, nevertheless, it has scarcely been applied for constrained evolutionary optimization. Recently, Peng *et al.* [48] took advantage of the Tchebycheff decomposition approach to solve COPs. In this method, N weight vectors are used to select N promising infeasible solutions, and the remaining $(NP - N)$ candidate solutions are selected based on the feasibility rule [49]. The parameter N is adjusted according to the feasible proportion. This method focuses on balancing diversity and convergence. However, another key issue of constrained evolutionary optimization, i.e., the tradeoff between constraints and objective function, is neglected to some degree. Besides, it only utilizes the Tchebycheff decomposition approach and the advantages of decomposition-based multiobjective optimization are not fully explored (such as the collaborative evolution of NP scalar optimization subproblems). The experimental results reveal that the performance of this method is limited on some complicated test functions from IEEE CEC2006 and IEEE CEC2010.

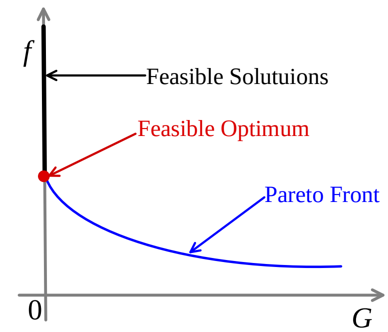


Fig. 1. Principle of $(f(\vec{x}), G(\vec{x}))$.

The above survey motivates us to further explore the potential of decomposition-based multiobjective optimization for solving COPs.

IV. Proposed Method

A. DeCODE

In DeCODE, a COP is transformed into the BOP $(f(\vec{x}), G(\vec{x}))$. The principle of this BOP is depicted in Fig. 1 [7], where the Pareto set is mapped to the Pareto front, all the feasible solutions are mapped to the solid segment, and the feasible optimum is mapped to the intersection of the f axis and the Pareto front. It is easy to derive that the search space S is mapped to points on and above the Pareto front.

DeCODE maintains a population of NP individuals, i.e., $P = \{\vec{x}_1, \ldots, \vec{x}_{NP}\}$, their objective function values, i.e., $\{f(\vec{x}_1), \ldots, f(\vec{x}_{NP})\}$, and their degree of constraint violation, i.e., $\{G(\vec{x}_1), \ldots, G(\vec{x}_{NP})\}$. The framework of DeCODE is described as follows.

Step 1) Initialization:

Step 1.1) For each $i \in \{1, \ldots, NP\}$, set $B_m(i) = \{1, \ldots, NP\}$, $B_r(i) = \{i\}$, and $flag = 0$.

Step 1.2) Initialize a set of NP weight vectors, i.e., $WV = \{(\lambda_1, 1-\lambda_1), \ldots, (\lambda_{NP}, 1-\lambda_{NP})\}$, where $\{\lambda_1, \ldots, \lambda_{NP}\}$ are uniformly generated between 0 and η, and initialize $\eta = 1$.

Step 1.3) Generate a random population with NP individuals: $P = \{\vec{x}_1, \ldots, \vec{x}_{NP}\}$, and evaluate P: $FV = \{(f(\vec{x}_1), G(\vec{x}_1)), \ldots, (f(\vec{x}_{NP}), G(\vec{x}_{NP}))\}$.

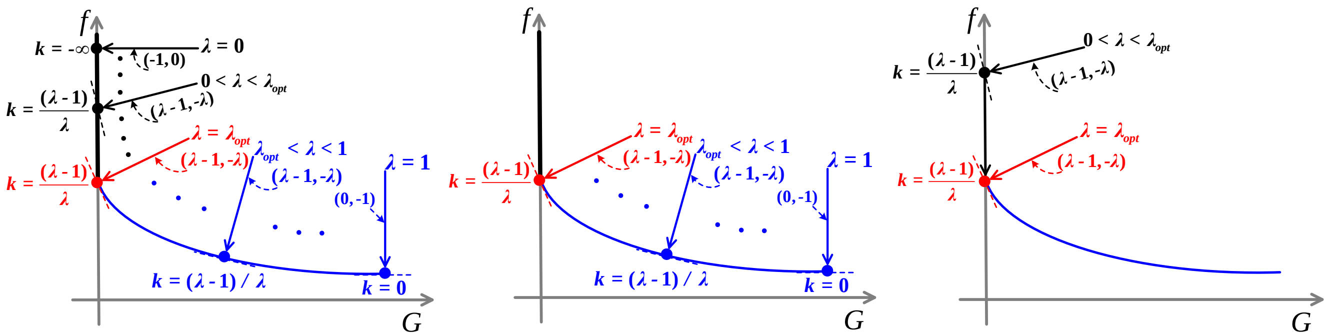
Step 2) Population updating:

Step 2.1) Generate an offspring population $OP = \{\vec{y}_1, \ldots, \vec{y}_{NP}\}$ by executing **the search algorithm**.

For $i = 1, \ldots, NP$ do

Step 2.2) Evaluate $\vec{y}_i$: $(f(\vec{y}_i), G(\vec{y}_i))$.

Step 2.3) For the index in $B_r(i)$ (i.e., i), if $g^{ws}(\vec{y}_i|(\lambda_i, 1-\lambda_i)) \leq g^{ws}(\vec{x}_i|(\lambda_i, 1-\lambda_i))$, set $\vec{x}_i = \vec{y}_i$, $f(\vec{x}_i) = f(\vec{y}_i)$, and $G(\vec{x}_i) = G(\vec{y}_i)$.



(a) $0 \leq \lambda \leq 1$ (b) $\lambda_{opt} < \lambda \leq 1$ (c) $0 < \lambda \leq \lambda_{opt}$

Fig. 2. Weight vectors with λ distributed between 0 and 1.

Step 3) Execute **the weight vector adjusting strategy** to adjust η adaptively, and generate a set of NP weight vectors, i.e., $\{(\lambda_1, 1-\lambda_1), \ldots, (\lambda_{NP}, 1-\lambda_{NP})\}$, by utilizing η.
Step 4) Executing **the restart strategy**.
Step 5) Stopping criterion: If the stopping criterion is satisfied, then stop and output the feasible solution with the smallest objective function value; otherwise, go to **Step 2)**.

As the above description, DeCODE shares the same framework with decomposition-based multiobjective optimization except for Step 3) and Step 4). Step 3) is designed to make decomposition-based framework suit the properties of COPs. In addition, Step 4) is developed to cope with extremely complicated constraints. Due to its numerous advantages such as ease of implementation, powerful search ability, and few algorithm-specific parameters, DE is employed to design the search algorithm in this paper. It is worth noting that prior to calculating the weighted sum of $\vec{x}_i$, its objective function value and degree of constraint violation are normalized as follows:

$$f^{norm}(\vec{x}_i) = \frac{f(\vec{x}_i) - f_{min}}{f_{max} - f_{min}} \tag{5}$$

$$G^{norm}(\vec{x}_i) = \frac{G(\vec{x}_i) - G_{min}}{G_{max} - G_{min}} \tag{6}$$

where f_{min} and f_{max} are the minimum and maximum objective function values in P, respectively, and G_{min} and G_{max} are the minimum and maximum degree of constraint violation in P, respectively.

Afterward, the weighted sum of $\vec{x}_i$ is calculated as follows:

$$g^{ws}(\vec{x}_i|(\lambda_i, 1-\lambda_i)) = \lambda_i f^{norm}(\vec{x}_i) + (1-\lambda_i)G^{norm}(\vec{x}_i) \tag{7}$$

where

$$\lambda_i = \frac{i}{NP} \cdot \eta \tag{8}$$

Remark 1: Compared with conventional mathematical programming methods, the advantages of DeCODE are summarized as follows:

1) Since DeCODE is population-based, it is more robust.
2) DeCODE does not impose strong assumptions such as linearity, convexity, and differentiability on objective function and constraints, which makes it applicable to diverse kinds of COPs.

Remark 2: Compared with other COEAs, the advantages of DeCODE are twofold:

1) Firstly, it shares the same framework with decomposition-based multiobjective optimization. Thus, the superiorities of the decomposition-based method (such as efficiency and collaborative evolution) can be inherited. Besides, the valuable knowledge developed for decomposition-based multiobjective optimization can be borrowed to further improve DeCODE.
2) Secondly, the weighted sum method is easy to implement. Moreover, it provides an effective way for constrained optimization. The reason is explained in the following. The aim of constrained optimization is to locate the feasible optimum on the Pareto front (as shown in Fig. 1), rather than a set of Pareto optimal solutions uniformly distributed on the whole Pareto front. Hence, the transformed BOP can be regarded as a BOP with a discrete Pareto optimal solution. As analyzed in [13], the weighted sum method is more effective than the Tchebycheff decomposition approach on the multiobjective 0-1 knapsack problem, which also has discrete Pareto optimal solutions.

In the following subsections, the weight vector adjusting strategy, the search algorithm, and the restart strategy are introduced sequentially.

B. Weight Vector Adjusting Strategy

The transformed BOP is optimized under the framework of decomposition-based multiobjective optimization. When a general BOP is solved, a set of representative solutions, the image of which is uniformly distributed on the whole Pareto front, is desired. Hence, a set of uniformly distributed weight vectors across the whole objective space is always maintained, where different weight vectors are expected to locate different points on the Pareto front. However, when the transformed BOP is solved, only the feasible optimum, which is the intersection of the f axis and the Pareto front, is wanted. This difference indicates that a set of weight vectors for seeking the whole Pareto front is not suitable for locating the single feasible optimum. To address this issue, a weight vector adjusting strategy is designed to generate proper weight vectors for locating the feasible optimum.

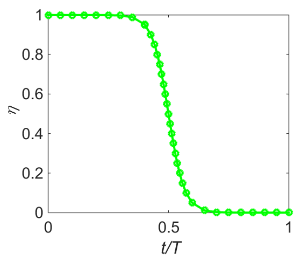


Fig. 3. Dynamic changing trajectory of η with $\alpha = 0.5$ and $\Gamma = 30$.

In multiobjective optimization, it is well-known that a Pareto optimal solution of a MOP, under mild conditions, is the optimal solution of the weighted sum scalar optimization subproblem with a weight vector $(\lambda, 1-\lambda)$ [13], [23]. Besides, as described in Fig. 2(a), the slope of the tangent at the image of this Pareto optimal solution in the objective space is $\frac{\lambda-1}{\lambda}$ and the corresponding direction vector is $(\lambda-1, -\lambda)$ [23]. Furthermore, if the Pareto front is convex and differentiable, the slope of the tangent will increase monotonously with the increase of G [50]. That is to say, the bigger the value of G of a Pareto optimal solution, the bigger the value of the slope $\frac{\lambda-1}{\lambda}$. Because $\frac{\lambda-1}{\lambda}$ increases monotonously with the increase of λ, it is easy to know that the bigger the value of G, the bigger the value of λ, and vice versa.

Assuming that $(\lambda_{opt}, 1-\lambda_{opt})$ is the weight vector attached to the feasible optimum, whose G is equal to 0, then the weight vector set $\{(\lambda, 1-\lambda)|0 < \lambda \leq 1\}$ can be divided into two subsets as shown in Fig. 2 (a), i.e., $\{(\lambda, 1-\lambda)|0 < \lambda \leq \lambda_{opt}\}$ and $\{(\lambda, 1-\lambda)|\lambda_{opt} < \lambda \leq 1\}$. The properties of these two subsets can be summarized as follows:

- As shown in Fig. 2(b), the weight vector $(\lambda, 1-\lambda)$ with $\lambda \in (\lambda_{opt}, 1]$ will locate a Pareto optimal solution with $G > 0$. That is to say, the weight vector with $\lambda \in (\lambda_{opt}, 1]$ cannot achieve the feasible optimum.
- As shown in Fig. 2(c), the weight vector $(\lambda, 1-\lambda)$ with $\lambda \in (0, \lambda_{opt}]$ will seek a feasible solution firstly. Subsequently, guided by objective function (i.e., the f axis), this feasible solution will approach the feasible optimum. To sum up, the weight vector with $\lambda \in (0, \lambda_{opt}]$ could achieve the feasible optimum finally.

Based on the above analysis, a set of weight vectors $(\lambda_i, 1-\lambda_i)(i \in \{1, \ldots, NP\})$, where λ_i is generated between 0 and λ_{opt}, would be helpful to achieve the feasible optimum. However, it is not easy to generate such a set of weight vectors accurately due to the fact that λ_{opt} is problem-dependent and cannot be known beforehand. In this paper, a simple yet effective method is proposed to approximate this set of weight vectors by decreasing the parameter η dynamically according to the famous sigmoid function, which has been widely employed in the community of evolutionary computation [19]:

$$\eta = \frac{1}{1 + e^{\Gamma(t/T-\alpha)}} \tag{9}$$

where t is the current generation number, T is the maximum generation number, and Γ and α are two critical parameters to control the decreasing trend of η. As shown in Fig. 3, η decreases in accordance with the sigmoid curve as the generation increases. At the early stage, η is very likely to be bigger than λ_{opt}. In this case, the weight vectors can be divided into two sets: one includes weight vectors with $\lambda \in (0, \lambda_{opt}]$, and the other contains weight vectors with $\lambda \in (\lambda_{opt}, 1]$. As discussed above, the first set of weight vectors can steer the solutions approaching the feasible optimum. Although the second set of weight vectors is not able to locate the feasible optimum directly, it can introduce the information of objective function, as shown in (7). Such information is beneficial to promote the exploration in the infeasible region [51], [52], [53]. At the later stage, η will be smaller than λ_{opt}. In this case, all of the generated weight vectors can motivate their solutions to find the feasible optimum. In summary, during the evolution, decreasing η based on (9) is a suitable way to generate a set of weight vectors, which has the potential to find the feasible optimum gradually.

Algorithm 1: Weight Vector Adjusting Strategy

Set $WV = \emptyset$;
if $flag == 0$ **then**
  if $\frac{t}{T} \leq p$ **then**
    $\varepsilon = \varepsilon_0(1 - \frac{t}{T})^{cp}$;
  else
    $\varepsilon = 0$;
  Calculate $FeaPro$ of the population;
  if $FeaPro \geq 0.85$ **then**
    $\varepsilon = 0$;
  if $G_{min} \geq \varepsilon$ **then**
    $flag = 1$;
    $\eta = 10^{-18}$;
  else
    $\eta = \frac{1}{1+e^{\Gamma(t/T-\alpha)}}$;
else
  $\eta = 10^{-18}$;
for $i = 1$ *to* NP **do**
  $\lambda_i = \frac{i}{NP} \cdot \eta$;
  $WV = WV \cup (\lambda_i, 1-\lambda_i)$;

As shown in (9) and Fig. 3, η decreases slowly at the early stage. When λ_{opt} of a COP is tiny, η may be larger than it for a relatively long period. Meanwhile, the number of weight vectors with $\lambda \in (\lambda_{opt}, \eta]$ would be much more than the number of weight vectors with $\lambda \in (0, \lambda_{opt}]$. Consequently, according to (7), much information of objective function will be used. Under this condition, much effort would be devoted to exploring the region around the Pareto front while neglecting the feasible optimum. To remedy this weakness, η should be truncated to a small value to suit λ_{opt}. As stated previously, we cannot know the value of λ_{opt} *a priori*, which signifies that we cannot know whether η needs to be truncated or not. Hence, a proper indicator should be used to reflect whether decreasing η according to (9) is suitable for the considered COP. Intuitively, if the decreasing manner of η is suitable for a COP, the degree of constraint violation would decrease consistently. Thus, we try to set a target level of degree of constraint violation at each generation. Once the target level cannot be satisfied, we consider that decreasing η according to (9) is not suitable. Under this condition, η is truncated to an extremely small value to guarantee that the number of weight vectors with $\lambda \in (0, \lambda_{opt}]$ is as many as possible. By this way, the feasible optimum could be achieved.

To set the target level at each generation, based on [51], the ε level controlling method is utilized here:

$$\varepsilon = \begin{cases} \varepsilon_0 (1 - \frac{t}{T})^{cp}, & \text{if } \frac{t}{T} \leq p \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

$$cp = -\frac{\log \varepsilon_0 + \beta}{\log(1-p)} \tag{11}$$

where ε_0 is the initial level, β is set to 6 in this paper, and p is an important parameter to control the target level at each generation. Similar to [51], in order to improve the usability, if the feasible proportion, i.e., $FeaPro$, exceeds FP (i.e., 0.85), ε is set to 0. As shown in (10), ε decreases with the increase of generation.

The whole process of the weight vector adjusting strategy is described in **Algorithm 1**. As shown in **Algorithm 1**, ε is the target level at each generation and $G_{min} \geq \varepsilon$ means that the target level cannot be fulfilled. Besides, η is is truncated to an extremely small value η_L (i.e., 10^{-18}) rather than 0. In this case, the information of objective function can be utilized to some extent.

C. Search Algorithm

When a search algorithm is designed for constrained optimization, it is expected to make a tradeoff not only between convergence and diversity, but also between constraints and objective function. In this paper, two DE trial vector generation strategies are integrated to achieve this goal, which are described below [54], [55].

- DE/rand-to-best/1/bin

$$\vec{v}_i = \vec{x}_{r_1} + F \cdot (\vec{x}_{best} - \vec{x}_{r_1}) + F \cdot (\vec{x}_{r_2} - \vec{x}_{r_3}) \tag{12}$$

$$u_{i,j} = \begin{cases} v_{i,j}, & \text{if } rand_j < CR \text{ or } j = j_{rand} \\ x_{i,j}, & \text{otherwise} \end{cases}, j = 1, \ldots, D \tag{13}$$

- DE/current-to-rand/1

$$\vec{u}_i = \vec{x}_i + rand \cdot (\vec{x}_{r_1} - \vec{x}_i) + F \cdot (\vec{x}_{r_2} - \vec{x}_{r_3}) \tag{14}$$

where $\vec{x}_i$, $\vec{v}_i$, and $\vec{u}_i$ are the ith target vector, the ith mutant vector, and the ith trial vector, respectively, $x_{i,j}$, $v_{i,j}$, and $u_{i,j}$ are the jth dimension of them, respectively, $\vec{x}_{r_1}$, $\vec{x}_{r_2}$, and $\vec{x}_{r_3}$ are three mutually different individuals randomly selected from the population, $\vec{x}_{best}$ is the individual with the best performance, F is the scaling factor, CR is the crossover control parameter, and j_{rand} is a random integer chosen from $\{1, \ldots, D\}$.

With respect to (12), the information of the best individual is utilized to generate a mutant vector. Consequently, the convergence can be accelerated via this strategy. Besides, in terms of (14), $\vec{x}_i$ learns the information of a randomly selected individual $\vec{x}_{r_1}$. Therefore, this strategy can promote the diversity.

These two strategies are combined in the following manner. For each individual, "DE/rand-to-best/1/bin" is executed with the probability $\frac{t}{T}$ while "DE/current-to-rand/1" is conducted with the probability $(1 - \frac{t}{T})$, where t and T are the current and maximum generation number, respectively. At the early stage, $\frac{t}{T}$ is small. So "DE/current-to-rand/1" will be used more frequently for exploration. At the later stage, $\frac{t}{T}$ becomes large. Thus, "DE/rand-to-best/1/bin" will be utilized more often for exploitation. By this manner, the tradeoff between convergence and diversity can be achieved.

Algorithm 2: Search Algorithm

Set $OP = \emptyset$;
for $i = 1$ *to* NP **do**
  Calculate $f^{norm}(\vec{x}_i)$ and $G^{norm}(\vec{x}_i)$ according to (5) and (6), respectively;
for $i = 1$ *to* NP **do**
  Randomly select a F value from the pool $\{0.6, 0.8, 1.0\}$;
  Randomly select a CR value from the pool $\{0.1, 0.2, 1.0\}$;
  if $rand < \frac{t}{T}$ **then**
    Set $WS_i = \emptyset$;
    for $j = 1$ *to* NP **do**
      $WS_i = WS_i \cup g^{ws}(\vec{x}_j | (\lambda_i, 1 - \lambda_i))$;
    Select $\vec{x}_{best}$ based on WS_i;
    Select $\vec{x}_{r_1}$, $\vec{x}_{r_2}$, and $\vec{x}_{r_3}$ from the population;
    Generate an offspring $\vec{u}_i$ according to (12) and (13);
  else
    Select $\vec{x}_{r_1}$, $\vec{x}_{r_2}$, and $\vec{x}_{r_3}$ from the population;
    Generate an offspring $\vec{u}_i$ according to (14);
  $OP = OP \cup \vec{u}_i$;

TABLE I
MAXIMUM NUMBER OF FUNCTION EVALUATIONS $MaxFEs$ AND POPULATION SIZE NP

Test Functions	$MaxFEs$	NP
24 test functions from IEEE CEC2006	5.0E+05	80
18 test functions with 10D from IEEE CEC2010	2.0E+05	60
18 test functions with 30D from IEEE CEC2010	6.0E+05	80
28 test functions with 50D from IEEE CEC2017	1.0E+06	100
28 test functions with 100D from IEEE CEC2017	2.0E+06	100

How to select the best individual in (12) has a direct effect on the tradeoff between constraints and objective function. In general, to achieve such a tradeoff, much information of objective function should be preferred at the early stage while little information of objective function should be favorable at the later stage. It is because much information of objective function is beneficial to promote the exploration in the infeasible region [51]–[53], while little information of objective function can promote the convergence to the feasible optimum. In this paper, the best individual is selected according to the weighted sum. Firstly, the normalized objective function value and degree of constraint violation are calculated for each individual according to (5) and (6), respectively. Afterward, a set of weighted sum is obtained on the basis of λ_i:

$$WS_i = \{g^{ws}(\vec{x}_1 | (\lambda_i, 1 - \lambda_i)), \ldots, g^{ws}(\vec{x}_{NP} | (\lambda_i, 1 - \lambda_i))\} \tag{15}$$

Finally, the individual with the minimum value in WS_i is selected as the best individual for $\vec{x}_i$. By doing this, each individual $\vec{x}_i$ can evolve along its own direction defined by the weight vector $(\lambda_i, 1 - \lambda_i)$. By making use of the weight vector adjusting strategy illustrated in Section IV-B, the information of objective function can be utilized properly and a tradeoff between constraints and objective function can be achieved.

Therefore, the above process is able to strike a tradeoff not only between convergence and diversity but also between constraints and objective function. In addition, two control parameters in DE, i.e., F and CR, are set in the same way as

TABLE II
EXPERIMENTAL RESULTS OF DECODE AND OTHER FOUR SELECTED METHODS OVER 25 INDEPENDENT RUNS ON 22 TEST FUNCTIONS FROM IEEE CEC2006

IEEE CEC2006	CMODE Mean OFV±Std Dev	NSES Mean OFV±Std Dev	DW Mean OFV±Std Dev	FROFI Mean OFV±Std Dev	DeCODE Mean OFV±Std Dev
g01	-1.5000E+01±0.00E+00*	-1.5000E+01±4.21E-30*	-1.5000E+01±5.02E-14*	-1.5000E+01±0.00E+00*	-1.5000E+01±0.00E+00*
g02	-8.0362E-01±2.42E-08*	-8.0362E-01±2.41E-32*	-8.0362E-01±9.99E-08*	-8.0362E-01±1.78E-07*	-8.0362E-01±3.12E-09*
g03	-1.0005E+00±5.29E-10*	-1.0005E+00±5.44E-19*	-1.0005E+00±4.27E-12*	-1.0005E+00±4.49E-16*	-1.0005E+00±4.00E-16*
g04	-3.0666E+04±2.64E-26*	-3.0666E+04±2.22E-24*	-3.066553E+04±0.00E+00*	-3.066553E+04±3.71E-12*	-3.066553E+04±3.71E-12*
g05	5.1265E+03±1.24E-27*	5.1265E+03±0.00E+00*	5.1264967E+03±4.22E-10*	5.1264967E+03±2.78E-12*	5.1265E+03±2.78E-12*
g06	-6.9618E+03±1.32E-26*	-6.9618E+03±0.00E+00*	-6.961813E+03±0.00E+00*	-6.961813E+03±0.00E+00*	-6.961813E+03±0.00E+00*
g07	2.4306E+01±7.65E-15*	2.4306E+01±7.37E-09*	2.430621E+01±5.28E-10*	2.430621E+01±6.32E-15*	2.4306E+01±8.52E-12*
g08	-9.5825E+02±6.36E-18*	-9.5825E+02±2.01E-34*	-9.5825E+02±2.78E-18*	-9.5825E+02±1.42E-17*	-9.5825E+02±1.42E-17*
g09	6.8063E+02±4.96E-14*	6.8063E+02±1.10E-25*	6.8063006E+02±2.23E-11*	6.8063006E+02±2.23E-11*	6.8063006E+02±2.54E-13*
g10	7.0492480E+03±2.52E-13*	7.0492480E+03±2.07E-24*	7.0492480E+03±4.43E-08*	7.0492480E+03±3.26E-12*	7.0492480E+03±6.34E-10*
g11	7.499E-01±0.00E+00*	7.499E-01±0.00E+00*	7.499E-01±1.06E-16*	7.499E-01±1.13E-16*	7.499E-01±1.13E-16*
g12	-1.00E+00±0.00E+00*	-1.00E+00±0.00E+00*	-1.00E+00±0.00E+00*	-1.00E+00±0.00E+00*	-1.00E+00±0.00E+00*
g13	5.3942E-02±1.04E-17*	5.3942E-02±1.98E-34*	5.3942E-02±6.03E-14*	5.3942E-02±2.41E-17*	5.3942E-02±2.13E-17*
g14	-4.776489E+01±3.62E-15*	-4.776489E+01±0.00E+00*	-4.776489E+01±3.47E-10*	-4.776489E+01±2.34E-14*	-4.776489E+01±2.93E-14*
g15	9.617150E+02±0.00E+00*	9.617150E+02±0.00E+00*	9.617150E+02±4.47E-13*	9.617150E+02±5.80E-13*	9.617150E+02±5.80E-13*
g16	-1.90516E+00±2.64E-26*	-1.90516E+00±2.62E-30*	-1.90516E+00±0.00E+00*	-1.90516E+00±4.53E-16*	-1.90516E+00±4.53E-16*
g17	8.853533E+03±1.24E-27*	8.853533E+03±2.51E-23*	8.880233E+03±3.63E+01	8.853533E+03±0.00E+00*	8.853533E+03±3.23E-08*
g18	-8.66025E-01±6.51E-17*	-8.66025E-01±4.62E-33*	-8.66025E-01±3.30E-07*	-8.66025E-01±6.94E-16*	-8.66025E-01±2.47E-16 *
g19	3.265559E+01±1.07E-10*	3.265559E+01±1.52E-05*	3.265559E+01±3.37E-07*	3.265559E+01±2.18E-14*	3.265559E+01±2.25E-14*
g21	1.937245E+02±5.34E+01*	1.937245E+02±1.62E-22*	1.937245E+02±3.66E-09*	1.937245E+02±2.95E-11*	1.937245E+02±4.82E-10*
g23	-4.000551E+02±7.33E-11*	-4.000551E+02±9.08E-26*	-4.000551E+02±6.49E-06*	-4.000551E+02±1.71E-13*	-4.000551E+02±1.66E-05*
g24	-5.50801E+00±8.24E-28*	-5.50801E+00±0.00E+00*	-5.50801E+00±0.00E+00*	-5.50801E+00±9.06E-16*	-5.50801E+00±9.06E-16*
*	**22**	**22**	**21**	**22**	**22**

in [54]. The details of the search algorithm are summarized in **Algorithm 2**.

D. Restart Strategy

In practice, some COPs may involve complicated constraints with strong nonlinearity and multimodality. Due to the complex infeasible region formed by these constraints, the population is very easy to stagnate. To address this issue, a restart strategy is introduced [55].

Before applying the restart strategy, one needs to answer a fundamental question: how to judge whether the population has already stagnated in the infeasible region or not. Intuitively, if the population converges to a small region in the infeasible region, the difference among the individuals will be tiny. Consequently, the individuals will have the similar degree of constraint violation or objective function values. Thus, we can conclude that the population has stagnated in the infeasible region when the following two conditions are satisfied:

1) All the individuals are infeasible.
2) All the individuals have the similar degree of constraint violation or objective function values, i.e., the standard deviation of the degree of constraint violation or objective function values is less than a predefined threshold μ.

Once these two conditions have been detected, the restart strategy will be triggered – all the solutions in the population will be regenerated from the decision space randomly. The reasons for regenerating the population randomly are twofold. Firstly, if the population has stagnated in the infeasible region, the information contained by the population is not useful for searching for the optimal solution. Secondly, a possible way to avoid stagnation is to exploit the feedback information from the evolution to guide the optimization. However, under this condition, more storage space is required. More importantly, it is not easy to decide what feedback information is promising.

Apparently, the threshold μ is critical to the restart strategy. A too big μ may lead to a wrong decision on the stagnation, which has a negative impact on convergence. On the contrary, a too small value cannot detect the stagnation timely, which would waste computational resources to some extent. Thus, it should be set carefully. We have investigated the setting of μ in the empirical study.

V. EMPIRICAL STUDY

A. Benchmark Test Functions and Parameter Settings

Three sets of benchmark test functions were employed to demonstrate the performance of DeCODE. The first set includes 24 test functions from IEEE CEC2006 [56], the second set contains 18 test functions with 10 dimensions (10D) and 30 dimensions (30D) from IEEE CEC2010 [57], and the third set involves 28 test functions with 50 dimensions (50D) and 100 dimensions (100D) from IEEE CEC2017 [58]. Note that these three sets of test functions exhibit various difficult properties, such as strong nonlinearity, tiny feasible region, and rotated landscape. Thus, they are able to provide a systematic assessment on the performance of DeCODE. More details about these three sets of test functions are referred to [56], [57], and [58].

The maximum number of functions evaluations (FEs) $MaxFEs$ and the population size NP are described in Table I. Note that NP is varied with different test sets and is related to the dimension of the search space. Following the suggestions in [56], [57], and [58], 25 independent runs were performed for each test function. In addition, the tolerance value δ for equality constraints was set to 10^{-4}. For DeCODE, ε_0 was set to min(ε_L, G_{max0}), where G_{max0} is the maximum degree of constraint violation in the initial population and $\varepsilon_L = 10^{D/2}$ is used to avoid a too large ε_0. Moreover, Γ in the sigmoid function, α in the sigmoid function, p in the ε level controlling method, and μ in the restart strategy were set to 30, 0.75, 0.85, and 10^{-6}, respectively.

TABLE III
EXPERIMENTAL RESULTS OF DECODE AND OTHER FIVE SELECTED METHODS OVER 25 INDEPENDENT RUNS ON 18 TEST FUNCTIONS WITH 10D FROM IEEE CEC2010

IEEE CEC2010 with 10D	ITLBO Mean OFV±Std Dev	FROFI Mean OFV±Std Dev	CACDE Mean OFV±Std Dev	AIS-IRP Mean OFV±Std Dev	DW Mean OFV±Std Dev	DeCODE Mean OFV±Std Dev
C01	-7.47E-01±1.87E-03+	-7.47E-01±1.35E-03+	-7.47E-01±1.88E-03+	-7.47E-01±1.30E-03+	-7.45E-01±3.66E-03−	-7.46E-01±5.02E-03
C02	-2.03E+00±8.14E-02−	-2.02E+00±1.41E-01−	-2.26E+00±6.57E-02+	-2.27E+00±2.00E-03+	-2.28E+00±2.46E-03+	-2.18E+00±1.27E-01
C03	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	3.75E-09±4.81E-04−	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C04	-1.00E-05±3.39E-15≈	-1.00E-05±0.00E+00≈	-1.00E-05±0.00E+00≈	-9.97E-06±4.28E-03−	-4.98E-03±7.63E-08+	-1.00E-05±8.42E-16
C05	-4.84E+02±1.11E-11≈	-4.84E+02±8.09E-07≈	-4.84E+02±3.48E-13≈	-4.80E+02±6.30E+00−	-4.84E+02±1.49E-07≈	-4.84E+02±3.48E-13
C06	-5.79E+02±2.39E-04≈	-5.79E+02±5.04E-04≈	-5.79E+02±1.68E-02≈	-5.80E+02±7.30E-08+	-5.80E+02±1.59E-03+	-5.79E+02±1.29E-13
C07	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	1.17E-08±2.70E+00−	4.65E+00±1.16E+00−	0.00E+00±0.00E+00
C08	8.47E+00±4.09E+00≈	7.11E+00±4.79E+00≈	7.01E+00±5.01E+00≈	4.09E+00±1.46E+00+	6.46E+00±5.06E+00+	8.56E+00±4.26E+00
C09	0.00E+00±0.00E+00+	2.50E+01±3.92E+01−	2.10E+01±3.51E+01−	2.70E+01±7.50E+01−	4.72E+00±8.38E-01≈	4.91E+00±1.82E+01
C10	1.92E-01±9.62E-01+	4.17E+01±8.69E-06≈	6.59E+01±4.40E+01−	1.62E+03±5.00E+02−	1.23E+01±1.82E+01+	4.17E+01±2.20E-14
C11	-1.51E-03±1.30E-05≈	-1.52E-03±5.63E-14≈	-1.52E-03±1.30E-06≈	-9.20E-04±8.23E-04−	∇−	-1.52E-03±3.77E-18
C12	-2.39E+01±1.14E+02+	-3.84E+02±2.17E+02+	-4.34E+02±2.49E+02+	-4.36E+02±6.02E+01+	-7.40E+01±2.51E+02+	-1.99E+00±4.81E-17
C13	-6.52E+01±1.78E+00−	-6.84E+01±2.52E-09≈	-6.72E+01±1.04E+00−	-6.79E+01±3.11E-01−	-6.56E+01±2.37E+00−	-6.84E+01±2.90E-14
C14	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	1.22E-04±2.90E-08−	4.45E+00±7.56E-01−	0.00E+00±0.00E+00
C15	3.54E+00±4.97E+00−	3.09E+00±1.37E+00−	3.38E+00±1.02E+00−	5.19E-09±1.10E-08+	3.67E+00±1.96E-10−	2.94E+00±1.50E+00
C16	2.27E-01±3.11E-01−	1.19E-02±2.07E-02−	4.52E-02±1.03E-01−	9.96E-18±6.27E-15−	2.96E-02±3.16E-02−	0.00E+00±0.00E+00
C17	3.91E-01±6.71E-01−	7.83E-02±2.25E-01−	1.23E-33±2.52E-33+	2.93E+00±2.29E+00−	4.79E-01±5.40E-01−	2.05E-11±4.44E-11
C18	0.00E+00±0.00E+00≈	5.23E-26±1.71E-25−	0.00E+00±0.00E+00≈	1.66E+00±1.27E+00−	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
−	**5**	**6**	**5**	**12**	**8**	/
+	**4**	**2**	**4**	**6**	**6**	/
≈	**9**	**10**	**9**	**0**	**4**	/

TABLE IV
RESULTS OF THE MULTIPLE-PROBLEM WILCOXON'S TEST FOR DECODE AND OTHER FIVE SELECTED METHODS ON 18 TEST FUNCTIONS WITH 10D FROM IEEE CEC2010

DeCoDE VS	R^+	R^-	p-value	α=0.1	α=0.05
ITLBO	102.5	68.5	≥ 0.2	No	No
FROFI	94.0	59.0	≥ 0.2	No	No
CACDE	87.5	82.5	≥ 0.2	No	No
AIS-IRP	124.0	47.0	1.30E-01	No	No
DW	114.0	57.0	≥ 0.2	No	No

TABLE V
RANKING OF DECODE AND OTHER FIVE SELECTED METHODS BY THE FRIEDMAN'S TEST ON 18 TEST FUNCTIONS WITH 10D FROM IEEE CEC2010

Algorithm	Ranking
DeCODE	**3.0389**
CACDE	3.1944
FROFI	3.3889
ITLBO	3.6389
DW	3.6944
AIS-IRP	3.9444

B. Experiments on the 24 Benchmark Test Functions from IEEE CEC2006

First, DeCODE was evaluated on the 24 benchmark test functions from IEEE CEC2006. Its performance was compared with four state-of-the-art COEAs with various constraint-handling techniques: CMODE [7], NSES [9], DW [48], and FROFI [54]. Note that CMODE and NSES are methods based on Pareto dominance. It can be known from [56] that it is extremely difficult to locate the optimum of g22 and there are no feasible solutions for g20. Thus, these two test functions were not considered here. The experimental results over 25 independent runs are summarized in Table II, where "Mean OFV" and "Std Dev" denote the average and standard deviation of objective function values over 25 runs, respectively. Due to the fact that the true optimum of each test function has been provided in [56], we can define a successful run as follows. A run for a test function is successful, if and only if $f(\vec{x}_{best}) - f(\vec{x}^*) < 10^{-4}$, where $\vec{x}^*$ is the optimum provided in [56] and $\vec{x}_{best}$ is the best feasible solution provided by a method. In Table II, "*" denotes that a method can satisfy the successful condition over all 25 runs for a test function.

It can be seen from Table II that among the five compared methods, CMODE, NSES, FROFI, and DeCODE can successfully obtain the optima of all test functions. However, DW cannot find the optimum of g17 consistently. In summary, the experimental results validate that DeCODE yields better or similar performance compared with other four competitors on the 24 test functions from IEEE CEC2006.

C. Experiments on the 18 Benchmark Test Functions with 10D and 30D from IEEE CEC2010

Subsequently, 36 complicated test functions from IEEE CEC2010 were taken into account. Due to the fact that the optimal solutions of these test functions are unknown, the average and standard deviation of objective function values over 25 runs were taken as the comparison criteria. Five state-of-the-art methods with various constraint-handling techniques were selected as the competitors: ITLBO [59], FROFI [54], CACDE [60], AIS-IRP [61], and DW [48]. The experimental results of ITLBO and FROFI can be available from our previous study. So the Wilcoxon's rank sum test at a 0.05 significance level was used to compare DeCODE with each of ITLBO and FROFI. We can just obtain the average and standard deviation of objective function values of CACDE, AIS-IRP, and DW from their original papers. Thus, the t-test at a 0.05 significance level was adopted to compare each of them with DeCODE. Furthermore, to compare these six methods simultaneously, the multiple-problem Wilcoxon's test and the Friedman's test were implemented via KEEL software [62]. Note that the Bonferroni-Dunn method was selected as the *post-hoc* method of the Friedman's test.

Regarding the test functions with 10D, the average and standard deviation of objective function values over 25 runs,

TABLE VI
EXPERIMENTAL RESULTS OF DECODE AND OTHER FIVE SELECTED METHODS OVER 25 INDEPENDENT RUNS ON 18 TEST FUNCTIONS WITH 30D FROM IEEE CEC2010

IEEE CEC2010 with 30D	ITLBO Mean OFV±Std Dev	FROFI Mean OFV±Std Dev	CACDE Mean OFV±Std Dev	AIS-IRP Mean OFV±Std Dev	CMODE Mean OFV±Std Dev	DeCODE Mean OFV±Std Dev
C01	-8.20E-01±8.95E-04≈	-8.21E-01±2.36E-03≈	-8.20E-01±2.67E-03≈	-8.20E-01±3.25E-04≈	-8.21E-01±3.3E-03≈	-8.19E-01±3.20E-03
C02	-2.03E+00±7.64E-02−	-2.00E+00±4.35E-02−	-2.01E+00±7.78E-02−	-2.21E+00±2.84E-03≈	9.75E-01±6.25E+01−	-2.23E+00±3.49E-02
C03	7.84E+01±6.31E+01−	2.87E+01±6.24E-08−	3.08E+01±3.50E+01−	6.68E+01±4.26E+02−	2.18E+01±1.25E+01≈	2.06E+01±1.31E+01
C04	1.69E-03±1.14E-03−	-3.33E-06±4.13E-10≈	3.54E+00±7.62E+00−	1.98E-03±1.61E-03−	6.72E-04±4.24E-04−	-3.33E-06±6.92E-12
C05	-4.82E+02±1.73E+00≈	-4.81E+02±2.84E+00≈	-3.41E+02±8.69E+01−	-4.36E+02±2.51E+01−	2.77E+02±2.03E+02∇−	-4.83E+02±1.53E-01
C06	-5.30E+02±4.80E-01≈	-5.29E+02±5.71E-01≈	-5.22E+02±2.92E+00−	-4.54E+02±4.79E+01−	-4.96E+02±2.15E+02∇−	-5.28E+02±1.46E+00
C07	1.59E-01±7.97E-01−	0.00E+00±0.00E+00≈	9.57E-01±1.74E+00−	1.07E+00±1.61E+00−	5.24E-05±5.89E-05−	0.00E+00±0.00E+00
C08	1.14E+01±2.79E+01−	0.00E+00±0.00E+00≈	9.76E+00±3.20E+01−	1.65E+00±6.41E-01−	3.68E-01±2.62E-01−	0.00E+00±0.00E+00
C09	2.86E+00±1.43E+01≈	4.30E+01±3.27E+01−	9.23E+03±1.26E+04−	1.57E+00±1.96E+00+	1.72E+13±1.07E+13∇−	8.97E+00±2.32E+01
C10	3.29E+01±1.41E+01≈	3.13E+01±8.22E-02≈	8.20E+10±3.91E+11−	1.78E+01±1.88E+01+	1.60E+13±7.00E+12∇−	3.13E+01±1.72E-05
C11	-3.86E-04±1.14E-05−	-3.92E-04±2.64E-06≈	2.99E-03±7.14E-03−	-1.58E-04±4.67E-05−	9.5E-03±9.7E-03∇−	-3.92E-04±3.11E-10
C12	-1.98E-01±2.39E-03≈	-1.99E-01±1.42E-06≈	-1.99E-01±2.35E-04≈	4.29E-06±4.52E-04−	-3.46E+00±7.35E+02∇−	-1.99E-01±1.23E-06
C13	-5.05E+01±1.18E+00−	-6.83E+01±1.95E-01≈	-6.77E+01±6.88E-01≈	-6.62E+01±2.27E-01−	-3.89E+01±2.17E+00−	-6.73E+01±1.60E+00
C14	4.78E-01±1.32E+00−	9.80E-29±4.90E-28≈	7.37E-26±1.79E-25≈	8.68E-07±3.14E-07−	9.31E+00±2.46E+00−	0.00E+00±0.00E+00
C15	2.38E+01±2.51E+01≈	2.16E+01±8.03E-05≈	2.17E+01±2.45E-01≈	3.41E+01±3.82E+01−	1.51E+13±8.26E+12−	2.18E+01±1.14E+00
C16	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	6.03E-04±3.02E-03−	8.21E-02±1.12E-01−	6.30E-02±2.72E-02−	0.00E+00±0.00E+00
C17	9.65E-01±1.73E+00−	1.59E-01±3.82E-01−	8.24E-01±6.85E-01−	3.61E+00±2.54E+00−	3.12E+02±2.75E+02∇−	4.48E-02±1.21E-01
C18	9.07E-17±3.18E-16+	4.87E-01±1.25E+00−	2.35E-05±8.46E-05−	4.02E+01±1.80E+01−	7.36E+03±3.12E+03−	3.03E-06±1.29E-05
−	**9**	**5**	**13**	**14**	**16**	/
+	**1**	**0**	**0**	**2**	**0**	/
≈	**8**	**13**	**5**	**2**	**2**	/

TABLE VII
RESULTS OF THE MULTIPLE-PROBLEM WILCOXON'S TEST FOR DECODE AND OTHER FIVE SELECTED METHODS ON 18 TEST FUNCTIONS WITH 30D FROM IEEE CEC2010

DeCODE VS	R^+	R^-	p-value	α=0.1	α=0.05
ITLBO	144.0	27.0	8.964E-03	Yes	Yes
FROFI	114.0	39.0	7.968E-02	Yes	No
CACDE	148.0	5.0	1.5258E-04	Yes	Yes
AIS-IRP	149.0	4.0	1.0682E-04	Yes	Yes
CMODE	153.0	0.0	1.5258E-05	Yes	Yes

TABLE VIII
RANKING OF DECODE AND OTHER FIVE SELECTED METHODS BY THE FRIEDMAN'S TEST ON 18 TEST FUNCTIONS WITH 30D FROM IEEE CEC2010

Algorithm	Ranking
DeCODE	**1.9444**
FROFI	2.5278
ITLBO	3.3889
CACDE	3.9722
AIS-IRP	4.0833
CMODE	5.0833

results of the multi-problem Wilcoxon's test, and results of the Friedman's test are summarized in Tables III, IV, and V, respectively. In Table III, "∇" denotes that feasible solutions cannot be found by a method consistently over 25 runs, and "−", "+", and "≈" represent that the performance of the corresponding competitor is worse than, better than, and similar to that of DeCODE in terms of the Wilcoxon's rank sum test/t-test, respectively. As shown in Table III, DeCODE performs better than ITLBO, FROFI, CACDE, AIS-IRP, and DW on five, six, five, 12, and eight test functions, respectively. In contrast, these five competitors outperform DeCODE on four, two, four, six, and six test functions, respectively. Note that DW cannot find any feasible solution on C11 and the experimental results of C11 are not provided in its original literature. In Table IV, all the R^+ values are bigger than the R^- values, which reflects that the performance of DeCODE is superior to that of other five competitors. Moreover, DeCODE achieves the first rank in the Friedman's test. Therefore, the experimental results demonstrate that DeCODE outperforms the five competitors on the 18 test functions with 10D from IEEE CEC2010.

In terms of the test functions with 30D, the average and standard deviation of objective function values over 25 runs, results of the multi-problem Wilcoxon's test, and results of the Friedman's test are reported in Tables VI, VII, and VIII, respectively. Note that, since the experimental results of DW on 30D cannot be obtained from the original paper [48], we removed DW and added CMODE [7] as a compared method. As shown in Table VI, DeCODE outperforms ITLBO, FROFI, CACDE, AIS-IRP, and CMODE on nine, five, 13, 14, and 16 test functions, respectively. However, ITLBO, FROFI, CACDE, AIS-IRP and CMODE cannot surpass DeCODE on more than two test functions. In Table VII, all the R^+ values are bigger than the R^- values, which reflects that the performance of DeCODE is better than that of the five competitors. Moreover, the significant difference at $\alpha = 0.1$ can be observed in all cases and the significant difference at $\alpha = 0.05$ can be found in four cases, i.e., DeCODE versus ITLBO, DeCODE versus CACDE, DeCODE versus AIS-IRP, and DeCODE versus CMODE. From Table VIII, DeCODE ranks the first according to the Friedman's test. Considering the experimental results, we can conclude that DeCODE has an edge over the five competitors on the 18 test functions with 30D from IEEE CEC2010.

To test the computational efficiency of DeCODE, its computational time was compared with CMODE, whose source code can be obtained online, on the 36 test functions from IEEE CEC2010. Note that CMODE is a Pareto dominance-based method. The experiments were performed on a computer with Intel Core (TM) i7-3770 (3.40 GHz) processor and Windows10 (64 bit) system. These two algorithms were programmed in MATLAB. The computational time provided by DeCODE is 139.55 seconds and 429.55 seconds for the 18 test functions with 10D and 30D over one run, respectively. The corresponding computational time resulting from CMODE is 200.84 seconds and 653.44 seconds, respectively. Thus, DeCODE is more efficient than CMODE, which also verifies that the decomposition-based framework is more efficient than nondominated sorting [13], [19].

In view of all the above experimental results, DeCODE shows overall better performance than the five competitors.

D. Experiments on the 28 Benchmark Test Functions with 50D and 100D from IEEE CEC2017

The 28 test functions with 50D and 100D from IEEE CEC2017 [58] were adopted to further evaluate DeCODE's performance on high-dimensional COPs. The two best algorithms, i.e., LSHADE44 [63] and UDE [64], in the IEEE CEC2017 competition were selected as the competitors. We compared DeCODE with each of LSHADE44 and UDE according to the ranking procedure provided in IEEE CEC2017:

- Rank the methods based on the feasible rate (*FR*), which denotes the percentage of runs where at least one feasible solution is found;
- Then rank the methods according to the average degree of constraint violation (*voi*);
- Finally, rank the methods in terms of the average objective function value.

To compare these three algorithms concurrently, we first ranked them on each test function according to the above procedure. Afterward, the total rank on all test functions was calculated. The experimental results are summarized in Table S-1 in the supplementary file. As shown in Table S-1, the total ranks of DeCODE on the 28 test functions with 50D and 100D are 45 and 44, respectively. Compared with DeCODE, the corresponding total ranks achieved by both LSHADE44 and UDE are larger. Therefore, DeCODE is better than LSHADE44 and UDE on these 56 test functions, which means that DeCODE has good scalability in solving high-dimensional COPs.

E. Effectiveness of the Weight Vector Adjusting Strategy

As introduced in Section IV-B, the weight vector adjusting strategy is used to generate proper weight vectors for locating the feasible optimum of a COP. To investigate the effectiveness of this strategy, five variants of DeCODE were implemented by setting η to five fixed values, i.e., $\eta = 0.1$, $\eta = 0.3$, $\eta = 0.5$, $\eta = 0.7$, and $\eta = 1.0$. The performance of DeCODE and these five variants was evaluated on the 18 test functions with 30D from IEEE CEC2010. Similar to Section V-C, the average and standard deviation of objective function values were recorded. In addition, if an algorithm fails to find at least one feasible solution consistently over 25 runs, the feasible rate was provided.

The Wilcoxon's rank sum test at a 0.05 significance level was used for performance comparison. The experimental results are summarized in Table S-2 in the supplementary file. As shown in Table S-2, DeCODE performs better than its five variants on 13, 14, 10, seven, and five test functions, respectively. However, these five variants cannot surpass DeCODE on more than one test function. The experimental results validate that the weight vector adjusting strategy plays a key role in making the decomposition-based framework suit the properties of COPs.

F. Effectiveness of the Search Algorithm

We implemented six variants of DeCODE where six different search algorithms were adopted. To be specific, in DeCODE-ConCon, the best individual in "DE/rand-to-best/1/bin" was selected based on $G(\vec{x})$. In DeCODE-ConObj, the best individual in "DE/rand-to-best/1/bin" was selected according to $f(\vec{x})$. In DeCODE-Con and DeCODE-Obj, only "DE/rand-to-best/1/bin" was used. Note that the best individual was selected based on $G(\vec{x})$ in DeCODE-Con and based on $f(\vec{x})$ in DeCODE-Obj, respectively. In DeCODE-Div, "DE/current-to-rand/1" was employed as the search algorithm while "DE/rand/1/bin" was used as the search algorithm in DeCODE-rand1. These six variants were evaluated on the 18 test functions with 30D from IEEE CEC2010. The experimental results are collected in Table S-3 in the supplementary file. Note that the performance of these six variants was compared with that of DeCODE based on the Wilcoxon's rank sum test at a 0.05 significance level.

From Table S-3, DeCODE outperforms the six variants on six, two, 14, eight, 18, and 16 test functions, respectively. However, no variant can provide better results on more than two test functions than DeCODE. By comparing DeCODE with DeCODE-ConCon and DeCODE-ConObj, it can be seen that properly utilizing $f(\vec{x})$/$G(\vec{x})$ is critical to a search algorithm. That is to say, the tradeoff between constraints and objective function is critical. By comparing DeCODE with DeCODE-Con, DeCODE-Obj, DeCODE-Div, and DeCODE-rand1, we can find that the tradeoff between diversity and convergence is also important for a search algorithm. In summary, the effectiveness of the proposed search algorithm has been confirmed.

G. Effectiveness of the Sigmoid Function

As described in Section IV-B, the sigmoid function controls the decreasing trend of η. As we know, the linear function and the exponential function are two other popular functions that can be used to control a dynamic parameter. To this end, we implemented two variants of DeCODE, i.e., DeCODE-Lin and DeCODE-Exp, which made use of the linear function (i.e., $\eta = 1 - \frac{t}{T}$) and the exponential function (i.e., $\eta = \frac{e^{30(1-\frac{t}{T})}-1}{e^{30}-1}$), respectively. The performance of DeCODE, DeCODE-Lin, and DeCODE-Exp was evaluated on the 36 test functions from IEEE CEC2010. The experimental results are summarized in Table S-4 in the supplementary file.

As shown in Table S-4, compared with DeCODE-Exp, DeCODE shows better performance on more test functions in terms of both 10D and 30D. Although DeCODE-Lin performs better than DeCODE on the 18 test functions with 10D, it is worse than DeCODE on the 18 test functions with 30D. Moreover, DeCODE-Lin cannot consistently find feasible solutions on C05 with 30D. Therefore, the experimental results verify the advantage of the sigmoid function.

TABLE IX
EXPERIMENTAL RESULTS OF DECODE AND DECODE-WOR OVER 25 INDEPENDENT RUNS ON THREE TEST FUNCTIONS

Instance	DeCODE Mean OFV±Std Dev (FR)	DeCODE-WoR Mean OFV±Std Dev (FR)
C11 with 30D	-3.92E-04±3.11E-10	(76%)
C12 with 30D	-1.99E-01±1.23E-06	(96%)
C17 with 30D	4.48E-02±1.21E-01	(80%)

H. Effectiveness of the Weighted Sum Method

We compared DeCODE with another variant, i.e., DeCODE-Tch, where the weighted sum method was replaced with the Tchebycheff decomposition approach. Both DeCODE and DeCODE-Tch were evaluated on the 36 test functions from IEEE CEC2010 and the experimental results are summarized in Table S-5 in the supplementary file. As shown in Table S-5, DeCODE-Tch cannot beat DeCODE on any test function while DeCODE provides better results on 10 test functions. The experimental results reflect the superiority of the weighted sum method for constrained optimization, which is in line with the analysis in Section IV-A.

I. Effectiveness of the Restart Strategy

In order to validate the effectiveness of the restart strategy, a competitor called DeCODE-WoR was implemented by removing the restart strategy from DeCODE. The 36 test functions from IEEE CEC2010 were used to produce the experimental results.

Similar to Section V-E, the average and standard deviation of objective function values and the feasible rate were recorded. Significant difference can be observed on three test functions, i.e., C11 with 30D, C12 with 30D, and C17 with 30D, based on the Wilcoxon's rank sum test at a 0.05 significance level. The experimental results of these three test functions are summarized in Table IX.

As shown in Table IX, on C11 with 30D, C12 with 30D, and C17 with 30D, DeCODE-WoR tends to be trapped in the infeasible region. Specifically, DeCODE-WoR converges to a local optimum in the infeasible region on these three test functions over six, one, and five runs, respectively.

In summary, the restart strategy can help the population jump out of the infeasible area where it has stagnated.

Remark 3: In Section S-I in the supplementary file, we also analyzed the effect of the parameter settings on the performance of DeCODE by extensive experiments.

VI. CONCLUSION

This paper further developed the potential of decomposition-based multiobjective optimization for constrained evolutionary optimization. In this paper, a COP was first transformed into a BOP. Thereafter, the transformed BOP was optimized under the decomposition-based framework. In order to make decomposition-based multiobjective optimization suit the properties of COPs, a weight vector adjusting strategy was proposed. In addition, DE was used to design the search algorithm. Moreover, a restart strategy was introduced to tackle COPs with complicated constraints. By the above process, an alternative COEA, i.e., DeCODE, was presented. Extensive and systematic experiments verified that:

1) The weight vector adjusting strategy is an effective way to adapt decomposition-based multiobjective optimization for COPs, by producing appropriate weight vectors.
2) The restart strategy improves DeCODE's ability to find feasible solutions for COPs with complicated constraints.
3) DeCODE shows superior performance against some state-of-the-art COEAs including Pareto dominance-based methods on three sets of benchmark test functions.

In the future, it is interesting to extend DeCODE to solve constrained multiobjective optimization problems. Note that, as an EA, the optimality and convergence of DeCODE cannot be theoretically guaranteed as conventional mathematical programming methods, especially in the scenarios which have high requirements for real-time performance and optimality [65], [66]. In the future, we will try to investigate COEAs from theoretical aspects.

The Matlab source code of DeCODE can be downloaded from Y. Wang's homepage: http://www.escience.cn/people/yongwang1/index.html

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Supplementary File for "Decomposition-based Multiobjective Optimization for Constrained Evolutionary Optimization"

S-1. Parameter Sensitivity Analysis

We investigated the sensitivities of all involved parameters experimentally. Extensive experiments were executed on the 18 test functions with 30D from IEEE CEC2010. Note that the comparisons were based on the Wilcoxon's rank sum test at a 0.05 significance level.

As shown in (9), Γ and α are two important parameters in the decreasing function of the weight vector adjusting strategy. In terms of α, we tested five different values in DeCODE, i.e., $\alpha = 0.0$, $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$, and $\alpha = 1.0$. The average and standard deviation of objective function values are collected in Table S-6 in the supplementary file. In addition, the feasible rate is provided if a method cannot achieve at least one feasible solution consistently over 25 runs. According to the collected results, DeCODE with $\alpha = 0.75$ outperforms DeCODE with $\alpha = 0.0$, $\alpha = 0.25$, $\alpha = 0.5$, and $\alpha = 1.0$ on 12, nine, four, and one test function, respectively. However, DeCODE with $\alpha = 0.0$, $\alpha = 0.25$, $\alpha = 0.5$, and $\alpha = 1.0$ cannot beat DeCODE with $\alpha = 0.75$ on any test function. In view of this, $\alpha = 0.75$ was recommended in this paper. Similarly, we implemented five variants of DeCODE, which had different Γ values, i.e., $\Gamma = 10$, $\Gamma = 20$, $\Gamma = 30$, $\Gamma = 40$, and $\Gamma = 50$. Note that $\Gamma = 30$ is used in the original DeCODE. Table S-7 in the supplementary file reports the experimental results. As shown in Table S-7, all variants perform similarly on most of the test functions. That is to say, this parameter is not sensitive. Overall, DeCODE with $\Gamma = 30$ exhibits better performance than other variants. Thus, $\Gamma = 30$ was employed in this paper.

As describe in (10) and (11), p and β are two important parameters in the ε level controlling method. We investigated their settings as follows. As far as p is concerned, we tested six different values in DeCODE, i.e., $p = 0.45$, $p = 0.55$, $p = 0.65$, $p = 0.75$, $p = 0.85$, and $p = 1$. The experimental results are summarized in Table S-8 in the supplementary file. It can be seen that DeCODE with $p = 0.85$ provides better results than DeCODE with $p = 0.45$, $p = 0.55$, $p = 0.65$, $p = 0.75$, and $p = 1.0$ on eight, six, four, one, and three test functions, respectively. However, the five competitors cannot outperform DeCODE with $p = 0.85$ on any test function. As a result, $p = 0.85$ was recommended for DeCODE. In terms of β, we tested five different values in DeCODE, i.e., $\beta = 0$, $\beta = 3$, $\beta = 6$, $\beta = 9$, and $\beta = 12$. Note that β is set to 6 in the original DeCODE. The experimental results are summarized in Table S-9 in the supplementary file. As shown in Table S-9, all variants perform similarly on most of the test functions. Thus, this parameter is not sensitive. DeCODE with $\beta = 6$ shows better performance on more test functions than other variants. Therefore, it was adopted in this paper.

Besides, two other parameters, i.e., ε_L and FP, which are also related to ε, were investigated experimentally. In terms of ε_L, it has little impact on test functions with 30D from IEEE CEC2010. Alternatively, test functions with 10D were adopted. We implemented five variants where five different ε_L values, i.e., $\varepsilon_L = 10^1$, $\varepsilon_L = 10^3$, $\varepsilon_L = 10^5$, $\varepsilon_L = 10^7$, and $\varepsilon_L = 10^9$, were used. Note that $\varepsilon_L = 10^5$ is used in the original DeCODE. The experimental results are collected in Table S-10 in the supplementary file. With respect to a too small ε_L (i.e., $\varepsilon_L = 10^1$ and $\varepsilon_L = 10^3$), the information of objective function cannot be utilized adequately. The performance of test functions needing the information of objective function, such as C02, C05, C06, C09, and C10, is affected. However, with respect to a too big ε_L (i.e., $\varepsilon_L = 10^9$), too much information of objective function has a negative impact on the convergence of some test functions. Overall, DeCODE with $\varepsilon_L = 10^5$ provides the best result. In terms of FP, we tested five different values, i.e., $FP = 0.55$, $FP = 0.65$, $FP = 0.75$, $FP = 0.85$, and $FP = 0.95$. Note that $FP = 0.85$ is utilized in the original DeCODE. The experimental results are summarized in Table S-11 in the supplementary file. As shown in Table S-11, all variants perform similarly on most of the test functions. Thus, FP is not sensitive. DeCODE with $FP = 0.85$ performs better on more test functions than other variants. Therefore, we made use of it in this paper.

In Section IV-B, η_L is a parameter to truncate η, which was set to a tiny value. We tested six different values to study its sensitivity, i.e., $\eta_L = 10^{-6}$, $\eta_L = 10^{-10}$, $\eta_L = 10^{-14}$, $\eta_L = 10^{-18}$, $\eta_L = 10^{-22}$, and $\eta_L = 10^{-26}$. The experimental results are reported in Table S-12 in the supplementary file. As shown in Table S-12, all variants provide similar results on most of test functions, thus η_L is insensitive.

μ is the key parameter in the restart strategy in Section IV-D. We implemented five variants of DeCODE where five different μ values, i.e., $\mu = 10^0$, $\mu = 10^{-3}$, $\mu = 10^{-6}$, $\mu = 10^{-9}$, and $\mu = 10^{-12}$, were utilized. The experimental results are summarized in Table S-13 in the supplementary file. It is clear that DeCODE with $\mu = 10^{-6}$ performs better than DeCODE with $\mu = 10^0$, $\mu = 10^{-3}$, $\mu = 10^{-9}$, and $\mu = 10^{-12}$ on 13, eight, two, and two test functions, respectively. However, the other four variants cannot be better than DeCODE with $\mu = 10^{-6}$ on more than one test function. As a consequence, $\mu = 10^{-6}$ was chosen in this paper.

TABLE S-1

EXPERIMENTAL RESULTS OF DECODE, LSHADE44, AND UDE OVER 25 INDEPENDENT RUNS ON THE 56 TEST FUNCTIONS FROM IEEE CEC2017

IEEE CEC2017	50D			100D		
	LSHADE44 Mean OFV/voi/FR (rank)	UDE Mean OFV/voi/FR (rank)	DeCODE Mean OFV/voi/FR (rank)	LSHADE44 Mean OFV/voi/FR (rank)	UDE Mean OFV/voi/FR (rank)	DeCODE Mean OFV/voi/FR (rank)
C01	7.79E-29/0.00E+00/1.00 **(1)**	3.18E-11/0.00E+00/1.00 **(1)**	5.42E-20/0.00E+00/1.00 **(1)**	1.03E-25/0.00E+00/1.00 **(1)**	1.79E-03/0.00E+00/1.00 **(3)**	3.36E-09/0.00E+00/1.00 **(1)**
C02	9.79E-29/0.00E+00/1.00 **(1)**	1.60E-11/0.00E+00/1.00 **(1)**	1.98E-20/0.00E+00/1.00 **(1)**	8.47E-26/0.00E+00/1.00 **(1)**	1.56E-03/0.00E+00/1.00 **(3)**	1.35E-09/0.00E+00/1.00 **(1)**
C03	8.95E+05/0.00E+00/1.00 **(3)**	1.09E+02/0.00E+00/1.00 **(2)**	8.64E-20/0.00E+00/1.00 **(1)**	2.73E+06/0.00E+00/1.00 **(3)**	7.42E+02/0.00E+00/1.00 **(2)**	2.31E-09/0.00E+00/1.00 **(1)**
C04	1.36E+01/0.00E+00/1.00 **(1)**	1.47E+02/0.00E+00/1.00 **(3)**	1.52E+01/0.00E+00/1.00 **(2)**	1.37E+01/0.00E+00/1.00 **(1)**	4.01E+02/0.00E+00/1.00 **(3)**	8.07E+01/0.00E+00/1.00 **(2)**
C05	1.68E-28/0.00E+00/1.00 **(1)**	1.34E+01/0.00E+00/1.00 **(3)**	6.38E-01/0.00E+00/1.00 **(2)**	3.28E-05/0.00E+00/1.00 **(1)**	7.54E+01/0.00E+00/1.00 **(3)**	2.99E+00/0.00E+00/1.00 **(2)**
C06	7.51E+03/1.17E-02/0.00 **(3)**	7.43E+02/0.00E+00/1.00 **(1)**	7.59E+02/0.00E+00/1.00 **(2)**	1.56E+04/9.81E-03/0.00 **(3)**	2.53E+03/4.28E-06/0.96 **(2)**	1.54E+03/0.00E+00/1.00 **(1)**
C07	-1.79E+02/0.00E+00/1.00 **(2)**	-9.78E+02/0.00E+00/1.00 **(1)**	-5.66E+01/0.00E+00/1.00 **(3)**	-3.02E+02/0.00E+00/1.00 **(2)**	-1.64E+03/0.00E+00/1.00 **(1)**	-1.43E+02/0.00E+00/1.00 **(3)**
C08	-1.30E-04/0.00E+00/1.00 **(1)**	1.45E-04/0.00E+00/1.00 **(3)**	-1.34E-04/0.00E+00/1.00 **(1)**	-4.81E-05/0.00E+00/1.00 **(1)**	2.97E-03/4.16E-06/0.92 **(3)**	1.57E-03/0.00E+00/1.00 **(2)**
C09	-2.04E-03/0.00E+00/1.00 **(1)**	-2.04E-03/0.00E+00/1.00 **(1)**	1.62E-02/0.00E+00/1.00 **(3)**	-1.43E-03/0.00E+00/1.00 **(1)**	2.46E-01/2.52E-24/0.84 **(3)**	6.86E-09/0.00E+00/1.00 **(2)**
C10	-4.83E-05/0.00E+00/1.00 **(1)**	3.04E-05/0.00E+00/1.00 **(3)**	-4.82E-05/0.00E+00/1.00 **(1)**	-1.72E-05/0.00E+00/1.00 **(1)**	5.57E-04/0.00E+00/1.00 **(3)**	3.58E-04/0.00E+00/1.00 **(2)**
C11	-1.76E+00/0.00E+00/1.00 **(1)**	-1.77E+02/4.36E-01/0.00 **(2)**	-4.49E+02/6.35E+03/0.00 **(3)**	-3.65E+00/5.25E-43/0.88 **(1)**	-1.84E+02/2.03E-01/0.00 **(3)**	-6.39E+03/1.41E-09/0.00 **(2)**
C12	4.98E+01/0.00E+00/1.00 **(3)**	2.09E+01/0.00E+00/1.00 **(2)**	1.28E+01/0.00E+00/1.00 **(1)**	3.25E+01/0.00E+00/1.00 **(3)**	1.07E+01/0.00E+00/1.00 **(1)**	2.49E+01/0.00E+00/1.00 **(2)**
C13	2.67E+01/0.00E+00/1.00 **(2)**	1.12E+03/0.00E+00/1.00 **(3)**	4.01E+00/0.00E+00/1.00 **(1)**	8.07E+01/0.00E+00/1.00 **(2)**	3.38E+04/2.69E+01/0.00 **(3)**	3.84E+01/0.00E+00/1.00 **(1)**
C14	1.40E+00/0.00E+00/1.00 **(3)**	1.23E+00/0.00E+00/1.00 **(2)**	1.10E+00/0.00E+00/1.00 **(1)**	9.72E-01/0.00E+00/1.00 **(3)**	9.14E-01/0.00E+00/1.00 **(1)**	9.28E-01/0.00E+00/1.00 **(2)**
C15	1.78E+01/0.00E+00/1.00 **(3)**	1.05E+01/0.00E+00/1.00 **(2)**	6.50E+00/0.00E+00/1.00 **(1)**	1.81E+01/0.00E+00/1.00 **(3)**	1.80E+01/0.00E+00/1.00 **(2)**	1.25E+01/0.00E+00/1.00 **(1)**
C16	2.53E+02/0.00E+00/1.00 **(3)**	1.21E+01/0.00E+00/1.00 **(2)**	6.28E+00/0.00E+00/1.00 **(1)**	5.35E+02/0.00E+00/1.00 **(3)**	3.37E+01/0.00E+00/1.00 **(2)**	6.79E+00/0.00E+00/1.00 **(1)**
C17	1.03E+00/2.55E+01/0.00 **(1)**	1.05E+00/2.55E+01/0.00 **(1)**	7.54E-01/2.55E+01/0.00 **(1)**	1.09E+00/5.05E+01/0.00 **(2)**	1.10E+00/5.05E+01/0.00 **(2)**	9.74E-01/4.95E+01/0.00 **(1)**
C18	5.67E+03/2.24E+05/0.00 **(1)**	4.06E+03/6.32E+07/0.00 **(2)**	4.13E+02/2.69E+10/0.00 **(3)**	3.44E+03/3.34E+06/0.00 **(1)**	8.33E+03/1.37E+08/0.00 **(2)**	9.58E+02/3.40E+08/0.00 **(3)**
C19	1.21E-05/3.61E+04/0.00 **(1)**	4.66E+00/3.61E+04/0.00 **(1)**	1.01E-03/3.61E+04/0.00 **(1)**	4.68E-05/7.30E+04/0.00 **(1)**	3.25E+01/7.30E+04/0.00 **(1)**	1.68E-03/7.30E+04/0.00 **(1)**
C20	3.20E+00/0.00E+00/1.00 **(1)**	7.59E+00/0.00E+00/1.00 **(3)**	3.88E+00/0.00E+00/1.00 **(2)**	9.36E+00/0.00E+00/1.00 **(2)**	1.89E+01/0.00E+00/1.00 **(3)**	8.23E+00/0.00E+00/1.00 **(1)**
C21	6.29E+01/0.00E+00/1.00 **(3)**	6.43E+00/0.00E+00/1.00 **(1)**	2.55E+01/0.00E+00/1.00 **(2)**	3.16E+01/0.00E+00/1.00 **(3)**	1.48E+01/0.00E+00/1.00 **(2)**	7.18E+00/0.00E+00/1.00 **(1)**
C22	8.39E+03/1.01E-02/0.96 **(2)**	2.90E+03/6.74E-02/0.84 **(3)**	1.70E+01/0.00E+00/1.00 **(1)**	5.04E+04/6.46E+00/0.04 **(2)**	5.58E+04/4.27E+02/0.00 **(3)**	3.61E+02/0.00E+00/1.00 **(1)**
C23	1.34E+00/0.00E+00/1.00 **(2)**	1.10E+00/0.00E+00/1.00 **(1)**	1.52E+00/0.00E+00/1.00 **(3)**	9.69E-01/0.00E+00/1.00 **(2)**	7.85E-01/0.00E+00/1.00 **(1)**	1.02E+00/0.00E+00/1.00 **(3)**
C24	1.43E+01/0.00E+00/1.00 **(3)**	1.13E+01/0.00E+00/1.00 **(2)**	5.62E+00/0.00E+00/1.00 **(1)**	1.72E+01/0.00E+00/1.00 **(2)**	1.81E+01/0.00E+00/1.00 **(3)**	5.75E+00/0.00E+00/1.00 **(1)**
C25	2.49E+02/0.00E+00/1.00 **(3)**	2.24E+01/0.00E+00/1.00 **(2)**	6.28E+00/0.00E+00/1.00 **(1)**	5.44E+02/0.00E+00/1.00 **(3)**	1.65E+02/0.00E+00/1.00 **(2)**	3.62E+01/0.00E+00/1.00 **(1)**
C26	1.04E+00/2.55E+01/0.00 **(1)**	1.05E+00/2.55E+01/0.00 **(1)**	9.62E-01/2.55E+01/0.00 **(1)**	1.10E+00/5.05E+01/0.00 **(2)**	1.10E+00/5.05E+01/0.00 **(2)**	1.03E+00/4.95E+01/0.00 **(1)**
C27	2.17E+04/1.34E+07/0.00 **(1)**	1.04E+04/2.58E+08/0.00 **(2)**	5.50E+02/4.73E+11/0.00 **(3)**	3.69E+04/4.78E+08/0.00 **(1)**	4.22E+04/2.03E+09/0.00 **(2)**	1.23E+03/3.80E+08/0.00 **(3)**
C28	2.65E+02/3.63E+04/0.00 **(1)**	1.25E+02/3.63E+04/0.00 **(1)**	1.53E+01/3.63E+04/0.00 **(1)**	5.84E+02/7.34E+04/0.00 **(2)**	3.20E+02/7.33E+04/0.00 **(2)**	1.58E+02/4.41E+03/0.00 **(1)**
Total Rank	**50**	**52**	**45**	**53**	**63**	**44**

TABLE S-2

EXPERIMENTAL RESULTS OF DECODE AND DECODE WITH FIVE FIXED η OVER 25 INDEPENDENT RUNS ON THE 18 TEST FUNCTIONS WITH 30D FROM IEEE CEC2010. FR DENOTES THE FEASIBLE RATE.

IEEE CEC2010 with 30D	$\eta = 0.1$ Mean OFV±Std Dev (FR)	$\eta = 0.3$ Mean OFV±Std Dev (FR)	$\eta = 0.5$ Mean OFV±Std Dev (FR)	$\eta = 0.7$ Mean OFV±Std Dev (FR)	$\eta = 1.0$ Mean OFV±Std Dev (FR)	DeCODE Mean OFV±Std Dev (FR)
C01	-5.78E-01±2.25E-02−	-5.75E-01±2.52E-02−	-5.79E-01±2.38E-02−	-5.80E-01±2.59E-02−	-5.09E-01±2.33E-02−	-8.19E-01±3.20E-03
C02	-7.62E-01±6.05E-01−	-1.98E+00±2.29E-01−	-2.26E+00±2.77E-02≈	-2.27E+00±2.20E-02≈	-2.27E+00±1.35E-02≈	-2.23E+00±3.49E-02
C03	2.21E+03±1.10E+04≈	1.26E+01±1.45E+01≈	1.15E+01±1.43E+01≈	1.61E+01±1.45E+01≈	(36%)−	2.06E+01±1.31E+01
C04	(96%)−	(92%)−	-3.33E-06±2.32E-11≈	-3.33E-06±3.38E-12≈	-3.33E-06±2.51E-12≈	-3.33E-06±6.92E-12
C05	(88%)−	-2.37E+02±1.23E+02−	(88%)−	(44%)−	-4.84E+02±4.16E-03≈	-4.83E+02±1.53E-01
C06	(76%)−	(76%)−	(76%)−	-5.16E+02±6.44E+00−	-5.29E+02±6.87E-01≈	-5.28E+02±1.46E+00
C07	1.46E-27±6.85E-27≈	1.59E-01±7.97E-01−	4.78E-01±1.32E+00−	3.19E-01±1.10E+00−	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C08	4.41E+00±2.12E+01−	3.67E+00±1.84E+01−	1.19E+01±3.31E+01−	1.59E-01±7.97E-01−	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C09	1.37E+10±3.93E+10−	8.94E+04±2.70E+05−	6.25E+01±6.68E+01−	3.55E+00±1.41E+01≈	5.28E+00±1.46E+00≈	8.97E+00±2.32E+01
C10	(84%)−	2.68E+05±9.35E+05−	3.39E+01±9.89E+00≈	3.13E+01±7.41E-03≈	3.14E+01±2.02E-02≈	3.13E+01±1.72E-05
C11	(0%)−	(0%)−	(0%)−	(0%)−	(0%)−	-3.92E-04±3.11E-10
C12	(48%)−	(0%)−	(0%)−	(0%)−	(0%)−	-1.99E-01±1.23E-06
C13	-6.80E+01±7.21E-01≈	-6.77E+01±1.43E+00≈	-6.83E+01±2.81E-01≈	-6.83E+01±3.38E-01≈	(0%)−	-6.73E+01±1.60E+00
C14	2.99E-11±6.98E-11−	3.19E-01±1.10E+00−	1.59E-01±7.97E-01−	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C15	1.01E+11±3.81E+11−	2.26E+01±2.05E+00≈	2.38E+01±1.09E+01≈	2.16E+01±1.50E-04≈	2.16E+01±1.16E-03≈	2.18E+01±1.14E+00
C16	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C17	6.65E-02±1.46E-01≈	1.64E-01±3.32E-01−	7.65E-02±1.81E-01≈	7.99E-02±1.70E-01≈	1.39E-02±3.41E-02≈	4.48E-02±1.21E-01
C18	5.80E+00±7.77E+00−	5.31E+00±2.11E+01−	7.69E+00±3.84E+01−	2.28E-20±1.14E-19+	8.44E-16±3.64E-15+	3.03E-06±1.29E-05
−	**13**	**14**	**10**	**7**	**5**	/
+	**0**	**0**	**0**	**1**	**1**	/
≈	**5**	**4**	**8**	**10**	**12**	/

TABLE S-3

EXPERIMENTAL RESULTS OF DECODE AND OTHER SIX VARIANTS WITH DIFFERENT SEARCH ALGORITHMS OVER 25 INDEPENDENT RUNS ON THE 18 TEST FUNCTIONS WITH 30D FROM IEEE CEC2010. FR DENOTES THE FEASIBLE RATE.

IEEE CEC2010 with 30D	DeCODE-ConCon Mean OFV±Std Dev (FR))	DeCODE-ConObj Mean OFV±Std Dev (FR)	DeCODE-Con Mean OFV±Std Dev (FR)	DeCODE-Obj Mean OFV±Std Dev (FR)	DeCODE-Div Mean OFV±Std Dev (FR)	DeCoDE-rand1 Mean OFV±Std Dev (FR)	DeCODE Mean OFV±Std Dev (FR)
C01	-8.19E-01±2.83E-03≈	-8.20E-01±2.11E-03≈	-8.14E-01±1.08E-02−	-8.18E-01±3.04E-03≈	-4.71E-01±1.85E-02−	-8.21E-01±1.36E-03≈	-8.19E-01±3.20E-03
C02	-2.17E+00±4.43E-02≈	-2.24E+00±2.48E-02≈	-2.02E+00±2.44E-01−	-2.21E+00±5.79E-02≈	-2.00E+00±1.90E-01−	-9.07E-01±1.34E-01−	-2.23E+00±3.49E-02
C03	1.49E+01±1.46E+01≈	1.84E+01±1.40E+01≈	1.17E+06±1.62E+06−	4.59E+00±1.07E+01+	(0%)−	2.87E+01±3.39E-06≈	2.06E+01±1.31E+01
C04	-3.33E-06±1.51E-09≈	-3.33E-06±5.00E-12≈	(20%)−	(64%)−	(0%)−	3.01E-03±9.52E-04−	-3.33E-06±6.92E-12
C05	(88%)−	-4.84E+02±2.81E-02≈	-2.24E+02±4.24E+01−	-4.84E+02±2.94E-13≈	(4%)−	(0%)−	-4.83E+02±1.53E-01
C06	-5.19E+02±4.04E+00−	-5.29E+02±8.54E-01≈	-2.49E+02±1.85E+02−	-5.31E+02±9.60E-02≈	(13%)−	(0%)−	-5.28E+02±1.46E+00
C07	0.00E+00±0.00E+00≈	5.43E-28±2.72E-27≈	4.78E-01±1.32E+00−	3.19E-01±1.10E+00−	1.71E+06±2.20E+06−	4.21E-02±5.21E-02−	0.00E+00±0.00E+00
C08	3.93E+00±1.96E+01−	0.00E+00±0.00E+00≈	9.35E+00±3.18E+01−	4.63E+00±1.85E+01−	1.53E+06±1.60E+06−	4.76E+00±1.09E+00−	0.00E+00±0.00E+00
C09	3.44E+01±3.41E+01−	5.71E+00±1.98E+01≈	1.71E+01±3.43E+01−	7.57E+00±2.14E+01≈	4.38E+08±3.98E+08−	(40%)−	8.97E+00±2.32E+01
C10	3.13E+01±4.55E-05≈	3.13E+01±1.87E-05≈	3.02E+01±9.76E+00≈	3.36E+01±9.00E+00≈	3.83E+08±5.70E+08−	(28%)−	3.13E+01±1.72E-05
C11	-3.92E-04±2.50E-10≈	-3.92E-04±1.47E-09≈	(24%)−	(0%)−	(0%)−	(0%)−	-3.92E-04±3.11E-10
C12	-1.99E-01±2.27E-08≈	(52%)−	(96%)−	(16%)−	(0%)−	(96%)−	-1.99E-01±1.23E-06
C13	-6.75E+01±1.27E+00≈	-6.78E+01±1.18E+00≈	-6.74E+01±8.72E-01≈	-6.81E+01±4.75E-01≈	-3.62E+01±2.88E+00−	-6.49E+01±9.87E-01−	-6.73E+01±1.60E+00
C14	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	3.19E-01±1.10E+00−	1.59E-01±7.97E-01−	7.83E+09±1.77E+10−	1.10E+01±8.83E-01−	0.00E+00±0.00E+00
C15	2.16E+01±7.21E-07≈	2.16E+01±6.04E-07≈	2.16E+01±5.71E-07≈	2.10E+01±4.52E+00≈	7.24E+09±6.51E+09−	1.94E+11±1.80E+11−	2.18E+01±1.14E+00
C16	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	1.05E-03±3.64E-03−	7.08E-02±4.02E-02−	2.54E-05±5.16E-05−	0.00E+00±0.00E+00
C17	1.28E-01±2.15E-01−	5.45E+00±2.51E+01−	5.29E-01±4.26E-01−	2.16E-01±1.99E-01−	(96%)−	5.56E+00±5.77E+00−	4.48E-02±1.21E-01
C18	6.69E+00±1.74E+01−	2.40E-17±9.60E-17+	1.29E+01±3.85E+01−	3.25E-22±1.55E-21+	3.63E+02±8.50E+02−	4.17E+03±3.21E+03−	3.03E-06±1.29E-05
−	**6**	**2**	**14**	**8**	**18**	**16**	/
+	**0**	**1**	**0**	**2**	**0**	**0**	/
≈	**12**	**15**	**4**	**8**	**0**	**2**	/

TABLE S-4
EXPERIMENTAL RESULTS OF DECODE, DECODE-LIN, AND DECODE-EXP OVER 25 INDEPENDENT RUNS ON THE 36 TEST FUNCTIONS FROM IEEE CEC2010. FR DENOTES THE FEASIBLE RATE.

IEEE CEC2010	10D			30D		
	DeCODE Mean OFV±Std Dev (FR)	DeCODE-Lin Mean OFV±Std Dev (FR)	DeCODE-Exp Mean OFV±Std Dev (FR)	DeCODE Mean OFV±Std Dev (FR)	DeCODE-Lin Mean OFV±Std Dev (FR)	DeCODE-Exp Mean OFV±Std Dev (FR)
C01	-7.46E-01±5.02E-03	-7.47E-01±1.35E-03+	-7.47E-01±1.87E-03+	-8.19E-01±3.20E-03	-8.19E-01±3.63E-03≈	-8.19E-01±2.69E-03≈
C02	-2.18E+00±1.27E-01	-2.27E+00±7.10E-03+	(88%)−	-2.23E+00±3.49E-02	-2.24E+00±3.29E-02≈	3.31E+00±8.30E-01−
C03	0.00E+00±0.00E+00	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	2.06E+01±1.31E+01	1.72E+01±1.43E+01≈	6.72E+14±1.12E+15−
C04	-1.00E-05±8.42E-16	-1.00E-05±0.00E+00≈	-1.00E-05±0.00E+00≈	-3.33E-06±6.92E-12	-3.33E-06±1.78E-11≈	7.79E-02±2.15E-01−
C05	-4.84E+02±3.48E-13	-4.84E+02±3.48E-13≈	(88%)−	-4.83E+02±1.53E-01	(96%)−	4.77E+02±5.25E+01−
C06	-5.79E+02±1.29E-13	-5.79E+02±1.29E-13≈	3.84E+02±1.78E+02−	-5.28E+02±1.46E+00	-5.07E+02±8.53E+00−	4.77E+02±9.01E+01−
C07	0.00E+00±0.00E+00	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00	3.19E-01±1.10E+00−	1.59E-01±7.97E-01−
C08	8.56E+00±4.26E+00	8.06E+00±4.54E+00≈	1.02E+01±2.25E+00−	0.00E+00±0.00E+00	1.06E+01±3.83E+01−	4.39E+00±2.20E+01−
C09	4.91E+00±1.82E+01	4.47E+00±1.60E+01≈	(64%)−	8.97E+00±2.32E+01	3.71E+02±4.60E+02−	4.72E+13±1.78E+13−
C10	4.17E+01±2.20E-14	3.84E+01±1.16E+01≈	(76%)−	3.13E+01±1.72E-05	4.95E+01±6.43E+01≈	(88%)−
C11	-1.52E-03±3.77E-18	-1.52E-03±3.54E-18≈	-1.52E-03±3.76E-18≈	-3.92E-04±3.11E-10	-3.92E-04±3.09E-10≈	-3.92E-04±1.64E-09≈
C12	-1.99E+00±4.81E-17	-2.19E+00±4.67E+00+	-6.98E-01±2.49E+00≈	-1.99E-01±1.23E-06	-1.99E-01±5.16E-09≈	-1.99E-01±1.46E-09≈
C13	-6.84E+01±2.90E-14	-6.84E+01±2.90E-14≈	-6.84E+01±2.97E-14≈	-6.73E+01±1.60E+00	-6.79E+01±7.60E-01≈	-6.76E+01±1.41E+00≈
C14	0.00E+00±0.00E+00	0.00E+00±0.00E+00≈	3.62E+03±1.80E+04−	0.00E+00±0.00E+00	0.00E+00±0.00E+00≈	4.86E-01±1.32E+00−
C15	2.94E+00±1.50E+00	2.06E+00±1.86E+00≈	4.95E+13±3.69E+13−	2.18E+01±1.14E+00	2.18E+01±1.14E+00≈	1.57E+14±6.46E+13−
C16	0.00E+00±0.00E+00	1.35E-03±4.73E-03−	5.33E-01±3.35E-01−	0.00E+00±0.00E+00	0.00E+00±0.00E+00≈	2.49E-01±2.68E-01−
C17	2.05E-11±4.44E-11	1.13E-11±3.99E-11≈	(92%)−	4.48E-02±1.21E-01	1.95E+01±9.73E+01−	(96%)−
C18	0.00E+00±0.00E+00	0.00E+00±0.00E+00≈	(96%)−	3.03E-06±1.29E-05	1.17E+00±3.60E+00−	2.71E+04±7.61E+03−
−	/	**1**	**10**	/	**7**	**4**
+	/	**3**	**1**	/	**0**	**0**
≈	/	**14**	**7**	/	**11**	**14**

TABLE S-5
EXPERIMENTAL RESULTS OF DECODE AND DECODE-TCH OVER 25 INDEPENDENT RUNS ON THE 36 TEST FUNCTIONS FROM IEEE CEC2010. FR DENOTES THE FEASIBLE RATE.

IEEE CEC2010	10D		30D	
	DeCODE Mean OFV±Std Dev (FR)	DeCODE-Tch Mean OFV±Std Dev (FR)	DeCODE Mean OFV±Std Dev (FR)	DeCODE-Tch Mean OFV±Std Dev (FR)
C01	-7.46E-01±5.02E-03	-7.46E-01±2.80E-03≈	-8.19E-01±3.20E-03	-8.18E-01±5.43E-03≈
C02	-2.18E+00±1.27E-01	(88%)−	-2.23E+00±3.49E-02	-2.21E+00±2.88E-02≈
C03	0.00E+00±0.00E+00	0.00E+00±0.00E+00≈	2.06E+01±1.31E+01	1.49E+01±1.46E+01≈
C04	-1.00E-05±8.42E-16	-1.00E-05±0.00E+00≈	-3.33E-06±6.92E-12	-3.33E-06±9.28E-12≈
C05	-4.84E+02±3.48E-13	-4.84E+02±3.48E-13≈	-4.83E+02±1.53E-01	-4.81E+02±1.19E+01≈
C06	-5.79E+02±1.29E-13	-5.79E+02±1.41E-13≈	-5.28E+02±1.46E+00	-5.27E+02±1.30E+00≈
C07	0.00E+00±0.00E+00	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00	6.38E-01±1.49E+00−
C08	8.56E+00±4.26E+00	9.22E+00±3.73E+00−	0.00E+00±0.00E+00	3.67E+00±1.84E+01−
C09	4.91E+00±1.82E+01	3.94E+00±1.61E+01≈	8.97E+00±2.32E+01	1.05E+01±2.00E+01≈
C10	4.17E+01±2.20E-14	4.17E+01±5.96E-11≈	3.13E+01±1.72E-05	3.13E+01±3.27E-05≈
C11	-1.52E-03±3.77E-18	-1.52E-03±3.29E-18≈	-3.92E-04±3.11E-10	-3.92E-04±1.24E-09≈
C12	-1.99E+00±4.81E-17	-6.30E-01±2.15E+00−	-1.99E-01±1.23E-06	(96%)−
C13	-6.84E+01±2.90E-14	-6.84E+01±2.90E-14≈	-6.73E+01±1.60E+00	-6.79E+01±7.57E-01≈
C14	0.00E+00±0.00E+00	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00	0.00E+00±0.00E+00≈
C15	2.94E+00±1.50E+00	3.38E+00±1.02E+00≈	2.18E+01±1.14E+00	2.16E+01±8.05E-07≈
C16	0.00E+00±0.00E+00	2.60E-03±6.23E-03−	0.00E+00±0.00E+00	0.00E+00±0.00E+00≈
C17	2.05E-11±4.44E-11	1.14E-10±1.65E-10−	4.48E-02±1.21E-01	7.60E+00±3.72E+01−
C18	0.00E+00±0.00E+00	0.00E+00±0.00E+00≈	3.03E-06±1.29E-05	2.62E-04±8.83E-04−
−	/	**5**	/	**5**
+	/	**0**	/	**0**
≈	/	**13**	/	**13**

TABLE S-6
EXPERIMENTAL RESULTS OF DECODE WITH FIVE VARYING α OVER 25 INDEPENDENT RUNS ON THE 18 TEST FUNCTIONS WITH 30D FROM IEEE CEC2010

IEEE CEC2010 with 30D	$\alpha = 0.0$ Mean OFV±Std Dev	$\alpha = 0.25$ Mean OFV±Std Dev	$\alpha = 0.5$ Mean OFV±Std Dev	$\alpha = 1.0$ Mean OFV±Std Dev	$\alpha = 0.75$ Mean OFV±Std Dev
C01	-8.18E-01±4.50E-03≈	-8.18E-01±5.85E-03≈	-8.19E-01±4.54E-03≈	-8.19E-01±3.61E-03≈	-8.19E-01±3.20E-03
C02	1.90E+00±1.26E+00−	-1.61E+00±5.62E-01−	-2.16E+00±8.40E-02≈	-2.24E+00±3.10E-02≈	-2.23E+00±3.49E-02
C03	4.61E+13±1.82E+12−	2.29E+01±1.17E+01−	2.87E+01±3.18E-08≈	1.95E+01±1.37E+01≈	2.06E+01±1.31E+01
C04	2.34E-01±3.18E-01−	-3.33E-06±1.22E-09≈	-3.33E-06±5.64E-12≈	-3.33E-06±5.64E-12≈	-3.33E-06±6.92E-12
C05	3.90E+02±5.34E+02−	-2.63E+02±2.33E+02−	-4.82E+02±6.97E-01≈	-4.84E+02±4.97E-02≈	-4.83E+02±1.53E-01
C06	4.12E+02±1.26E+02−	-4.35E+02±1.94E+02−	-5.26E+02±2.25E+00≈	-5.29E+02±8.69E-01≈	-5.28E+02±1.46E+00
C07	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C08	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C09	1.67E+13±1.11E+13−	4.32E+12±1.22E+13−	2.75E+03±7.30E+03−	7.04E+00±1.65E+01≈	8.97E+00±2.32E+01
C10	2.26E+13±1.23E+13−	1.81E+12±8.67E+12−	5.91E+01±5.56E+01−	3.13E+01±7.03E-04≈	3.13E+01±1.72E-05
C11	-3.92E-04±1.64E-07≈	-3.92E-04±4.21E-10≈	-3.92E-04±1.87E-10≈	-3.92E-04±1.39E-10≈	-3.92E-04±3.11E-10
C12	-1.99E-01±1.14E-16≈	-1.99E-01±7.02E-09≈	-1.99E-01±4.72E-04≈	-1.99E-01±2.06E-08≈	-1.99E-01±1.23E-06
C13	-6.78E+01±9.04E-01≈	-6.78E+01±1.09E+00≈	-6.79E+01±9.91E-01≈	-6.78E+01±8.80E-01≈	-6.73E+01±1.60E+00
C14	5.01E-08±2.35E-07−	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C15	3.01E+13±2.06E+13−	5.86E+07±2.45E+08−	2.16E+01±1.23E-06≈	2.16E+01±5.68E-07≈	2.18E+01±1.14E+00
C16	1.03E-02±1.82E-02−	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C17	2.12E+00±2.84E+00−	4.84E+00±1.77E+00−	8.00E-01±2.67E+00−	2.44E+04±1.21E+02−	4.48E-02±1.21E-01
C18	1.21E+04±7.56E+03−	1.92E+02±3.49E+02−	6.20E+00±1.87E+01−	1.13E-06±5.64E-06≈	3.03E-06±1.29E-05
−	**12**	**9**	**4**	**1**	**/**
+	**0**	**0**	**0**	**0**	**/**
≈	**6**	**9**	**14**	**17**	**/**

TABLE S-7
EXPERIMENTAL RESULTS OF DECODE WITH FIVE VARYING Γ OVER 25 INDEPENDENT RUNS ON THE 18 TEST FUNCTIONS WITH 30D FROM IEEE CEC2010

IEEE CEC2010 with 30D	$\Gamma = 10$ Mean OFV±Std Dev	$\Gamma = 20$ Mean OFV±Std Dev	$\Gamma = 40$ Mean OFV±Std Dev	$\Gamma = 50$ Mean OFV±Std Dev	$\Gamma = 30$ Mean OFV±Std Dev
C01	-8.19E-01±2.65E-03≈	-8.20E-01±3.09E-03≈	-8.19E-01±3.52E-03≈	-8.19E-01±3.57E-03≈	-8.19E-01±3.20E-03
C02	-2.27E+00±1.31E-02≈	-2.25E+00±1.92E-02≈	-2.24E+00±2.66E-02≈	-2.24E+00±2.16E-02≈	-2.23E+00±3.49E-02
C03	2.06E+01±1.31E+01≈	1.72E+01±1.43E+01≈	2.41E+01±1.07E+01≈	1.84E+01±1.40E+01≈	2.06E+01±1.31E+01
C04	-3.33E-06±5.91E-12≈	-3.33E-06±6.91E-12≈	-3.33E-06±6.23E-11≈	-3.33E-06±1.14E-11≈	-3.33E-06±6.92E-12
C05	-4.83E+02±2.07E-01≈	-4.83E+02±1.53E-01≈	-4.83E+02±1.51E-01≈	-4.83E+02±8.77E-02≈	-4.83E+02±1.53E-01
C06	-5.26E+02±1.87E+00≈	-5.27E+02±1.83E+00≈	-5.27E+02±1.67E+00≈	-5.27E+02±1.19E+00≈	-5.28E+02±1.46E+00
C07	3.19E-01±1.10E+00−	1.65E-27±4.56E-27≈	1.59E-01±7.97E-01−	1.59E-01±7.97E-01−	0.00E+00±0.00E+00
C08	8.95E+00±4.47E+01−	4.39E+00±2.20E+01−	1.95E+01±6.79E+01−	3.19E-01±1.10E+00−	0.00E+00±0.00E+00
C09	5.80E+00±1.91E+01≈	3.45E+00±1.46E+01≈	1.76E-01±8.80E-01+	5.28E-01±1.46E+00+	8.97E+00±2.32E+01
C10	3.06E+01±3.72E+00≈	3.13E+01±1.52E-05≈	3.13E+01±9.08E-06≈	3.13E+01±3.01E-05≈	3.13E+01±1.72E-05
C11	-3.92E-04±3.36E-08≈	-3.92E-04±1.74E-10≈	-3.92E-04±3.88E-10≈	-3.92E-04±2.75E-10≈	-3.92E-04±3.11E-10
C12	-1.99E-01±9.41E-08≈	-1.99E-01±1.51E-08≈	-1.99E-01±1.93E-08≈	-1.99E-01±8.17E-09≈	-1.99E-01±1.23E-06
C13	-6.79E+01±8.06E-01≈	-6.78E+01±1.02E+00≈	-6.78E+01±8.80E-01≈	-6.77E+01±1.02E+00≈	-6.73E+01±1.60E+00
C14	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C15	2.16E+01±4.36E-07≈	2.16E+01±3.82E-07≈	2.16E+01±1.22E-06≈	2.16E+01±9.40E-07≈	2.18E+01±1.14E+00
C16	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C17	9.05E-02±1.51E-01≈	8.56E-02±1.41E-01≈	8.46E-02±1.40E-01≈	6.37E-02±1.36E-01≈	4.48E-02±1.21E-01
C18	1.49E-01±7.45E-01−	3.01E-02±1.50E-01−	6.29E-07±3.14E-06+	9.80E-11±4.50E-10+	3.03E-06±1.29E-05
−	**3**	**2**	**2**	**2**	**/**
+	**0**	**0**	**2**	**2**	**/**
≈	**15**	**16**	**14**	**14**	**/**

TABLE S-8
EXPERIMENTAL RESULTS OF DECODE WITH SIX VARYING p OVER 25 INDEPENDENT RUNS ON THE 18 TEST FUNCTIONS WITH 30D FROM IEEE CEC2010. FR DENOTES THE FEASIBLE RATE.

IEEE CEC2010 with 30D	$p = 0.45$ Mean OFV±Std Dev (FR)	$p = 0.55$ Mean OFV±Std Dev (FR)	$p = 0.65$ Mean OFV±Std Dev (FR)	$p = 0.75$ Mean OFV±Std Dev (FR)	$p = 1.0$ Mean OFV±Std Dev (FR)	$p = 0.85$ Mean OFV±Std Dev (FR)
C01	-8.19E-01±2.97E-03≈	-8.20E-01±2.15E-03≈	-8.18E-01±3.30E-03≈	-8.19E-01±3.37E-03≈	-8.19E-01±3.78E-03≈	-8.19E-01±3.20E-03
C02	-1.84E+00±2.14E-01−	-1.97E+00±1.10E-01−	-2.10E+00±9.56E-02−	-2.22E+00±4.60E-02≈	-2.25E+00±1.79E-02≈	-2.23E+00±3.49E-02
C03	1.72E+01±1.43E+01−	2.06E+01±1.31E+01≈	2.06E+01±1.31E+01≈	2.06E+01±1.31E+01≈	1.95E+01±1.37E+01≈	2.06E+01±1.31E+01
C04	-3.33E-06±5.05E-12≈	-3.33E-06±2.71E-11≈	-3.33E-06±7.47E-12≈	-3.33E-06±2.85E-12≈	-3.33E-06±2.22E-11≈	-3.33E-06±6.92E-12
C05	2.14E+02±3.53E+02−	-3.06E+02±3.04E+02−	-4.83E+02±2.44E-01≈	-4.83E+02±2.50E-01≈	-4.83E+02±1.60E-01≈	-4.83E+02±1.53E-01
C06	-4.56E+02±2.32E+02−	-5.27E+02±1.60E+00≈	-5.28E+02±1.27E+00≈	-5.28E+02±1.03E+00≈	-5.28E+02±1.32E+00≈	-5.28E+02±1.46E+00
C07	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C08	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C09	(72%)−	5.89E+02±1.06E+03−	2.28E+01±3.10E+01−	1.92E+01±4.30E+01−	7.83E+00±1.98E+01≈	8.97E+00±2.32E+01
C10	(96%)−	1.12E+03±5.45E+03−	3.13E+01±6.91E-06≈	3.13E+01±4.71E-05≈	3.13E+01±1.32E-05≈	3.13E+01±1.72E-05
C11	-3.92E-04±4.71E-09≈	-3.92E-04±1.71E-10≈	-3.92E-04±4.05E-10≈	-3.92E-04±6.15E-09≈	(0%)−	-3.92E-04±3.11E-10
C12	-1.99E-01±1.32E-14≈	-1.99E-01±1.04E-07≈	-1.99E-01±1.03E-04≈	-1.99E-01±2.13E-04≈	(0%)−	-1.99E-01±1.23E-06
C13	-6.77E+01±8.87E-01≈	-6.76E+01±1.19E+00≈	-6.77E+01±1.05E+00≈	-6.79E+01±8.82E-01≈	-6.70E+01±1.40E+00≈	-6.73E+01±1.60E+00
C14	0.00E+00±0.00E+00≈	5.57E-29±2.78E-28≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C15	2.16E+01±4.51E-07≈	2.16E+01±3.84E-07≈	2.16E+01±5.72E-07≈	2.16E+01±5.59E-07≈	2.16E+01±9.77E-07≈	2.18E+01±1.14E+00
C16	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C17	4.26E+01±1.66E+02−	7.56E+00±3.73E+01−	2.49E+01±1.24E+02−	8.93E-02±1.69E-01≈	1.37E-01±1.69E-01−	4.48E-02±1.21E-01
C18	1.07E+02±1.97E+02−	6.99E+00±3.49E+01−	2.16E+00±7.72E+00−	1.04E-05±5.10E-05≈	9.78E-05±3.39E-04≈	3.03E-06±1.29E-05
−	**8**	**6**	**4**	**1**	**3**	**/**
+	**0**	**0**	**0**	**0**	**0**	**/**
≈	**10**	**12**	**14**	**17**	**15**	**/**

TABLE S-9
EXPERIMENTAL RESULTS OF DECODE WITH FIVE VARYING β OVER 25 INDEPENDENT RUNS ON THE 18 TEST FUNCTIONS WITH 30D FROM IEEE CEC2010. FR DENOTES THE FEASIBLE RATE.

IEEE CEC2010 with 30D	$\beta = 0$ Mean OFV±Std Dev (FR)	$\beta = 3$ Mean OFV±Std Dev (FR)	$\beta = 9$ Mean OFV±Std Dev (FR)	$\beta = 12$ Mean OFV±Std Dev (FR)	$\beta = 6$ Mean OFV±Std Dev (FR)
C01	-8.18E-01±5.13E-03≈	-8.19E-01±3.76E-03≈	-8.19E-01±3.39E-03≈	-8.18E-01±3.72E-03≈	-8.19E-01±3.20E-03
C02	-2.24E+00±1.73E-02≈	-2.24E+00±2.45E-02≈	-2.24E+00±2.34E-02≈	-2.23E+00±3.57E-02≈	-2.23E+00±3.49E-02
C03	2.29E+01±1.17E+01≈	1.84E+01±1.40E+01≈	1.72E+01±1.43E+01≈	2.18E+01±1.25E+01≈	2.06E+01±1.31E+01
C04	-3.33E-06±4.72E-12≈	-3.33E-06±5.54E-11≈	-3.33E-06±8.50E-11≈	-3.33E-06±1.42E-11≈	-3.33E-06±6.92E-12
C05	-4.83E+02±1.50E-01≈	-4.83E+02±3.27E-01≈	-4.83E+02±2.26E-01≈	-4.82E+02±9.16E+00≈	-4.83E+02±1.53E-01
C06	-5.27E+02±1.48E+00≈	-5.27E+02±1.73E+00≈	-5.28E+02±1.12E+00≈	-5.27E+02±1.72E+00≈	-5.28E+02±1.46E+00
C07	3.19E-01±1.10E+00−	1.59E-01±7.97E-01−	1.59E-01±7.97E-01−	6.29E-28±2.73E-27≈	0.00E+00±0.00E+00
C08	5.43E-28±2.72E-27≈	5.43E-28±2.72E-27≈	3.93E+00±1.96E+01−	4.72E+00±2.36E+01−	0.00E+00±0.00E+00
C09	3.59E+00±1.44E+01≈	4.68E+00±1.64E+01≈	1.09E+01±3.89E+01≈	5.28E-01±1.46E+00+	8.97E+00±2.32E+01
C10	3.13E+01±9.49E-06≈	3.13E+01±5.71E-05≈	3.13E+01±2.68E-05≈	3.13E+01±1.61E-05≈	3.13E+01±1.72E-05
C11	-3.92E-04±6.29E-10≈	-3.92E-04±9.36E-08≈	-3.92E-04±8.10E-09≈	-3.92E-04±4.61E-10≈	-3.92E-04±3.11E-10
C12	-1.99E-01±3.23E-06≈	-1.99E-01±7.10E-09≈	(96%)−	(96%)−	-1.99E-01±1.23E-06
C13	-6.76E+01±1.14E+00≈	-6.75E+01±1.58E+00≈	-6.77E+01±1.13E+00≈	-6.73E+01±1.39E+00≈	-6.73E+01±1.60E+00
C14	0.00E+00±0.00E+00≈	5.57E-29±2.78E-28≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C15	2.16E+01±5.01E-07≈	2.16E+01±1.15E-06≈	2.18E+01±1.14E+00≈	2.16E+01±3.07E-07≈	2.18E+01±1.14E+00
C16	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C17	9.34E-02±1.55E-01≈	7.74E-02±1.42E-01≈	1.44E-01±1.70E-01−	6.26E-02±1.31E-01−	4.48E-02±1.21E-01
C18	5.38E-1±2.69E-0−	5.70E-01±2.85E+00−	1.10E-14±5.40E-14+	8.97E-08±4.49E-07+	3.03E-06±1.29E-05
−	**2**	**2**	**4**	**3**	**/**
+	**0**	**0**	**1**	**2**	**/**
≈	**16**	**16**	**13**	**13**	**/**

TABLE S-10
EXPERIMENTAL RESULTS OF DECODE WITH FIVE VARYING ε_L OVER 25 INDEPENDENT RUNS ON THE 18 TEST FUNCTIONS WITH 10D FROM IEEE CEC2010. FR DENOTES THE FEASIBLE RATE.

IEEE CEC2010 with 10D	$\varepsilon_L = 10^1$ Mean OFV±Std Dev (FR)	$\varepsilon_L = 10^3$ Mean OFV±Std Dev (FR)	$\varepsilon_L = 10^7$ Mean OFV±Std Dev (FR)	$\varepsilon_L = 10^9$ Mean OFV±Std Dev (FR)	$\varepsilon_L = 10^5$ Mean OFV±Std Dev (FR)
C01	-7.47E-01±1.87E-03+	-7.47E-01±2.25E-03+	-7.46E-01±1.35E-03≈	-7.47E-06±1.87E-03≈	-7.46E-01±5.02E-03
C02	-1.96E+00±4.06E-01−	(96%)−	-2.13E+00±1.92E-01≈	-1.91E+00±1.09E+00−	-2.18E+00±1.27E-01
C03	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	4.62E+00±4.53E+00−	5.33E+00±4.44E+00−	0.00E+00±0.00E+00
C04	-1.00E-05±0.00E+00≈	-1.00E-05±0.00E+00≈	-1.00E-05±9.80E-12≈	2.71E-02±1.36E-01−	-1.00E-05±8.42E-16
C05	(92%)−	-4.84E+02±3.48E-13≈	-4.84E+02±3.48E-13≈	-4.84E+02±3.48E-13≈	-4.84E+02±3.48E-13
C06	3.93E+02±1.82E+02−	-5.79E+02±1.23E-13≈	-5.79E+02±1.29E-13≈	-5.79E+02±1.23E-13≈	-5.79E+02±1.29E-13
C07	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C08	9.04E+00±4.03E+00≈	7.79E+00±4.57E+00≈	7.95E+00±4.76E+00≈	9.47E+00±3.57E+00≈	8.56E+00±4.26E+00
C09	(84%)−	9.94E+00±2.53E+01≈	9.82E+00±2.61E+01≈	3.70E+00±1.49E+01≈	4.91E+00±1.82E+01
C10	(84%)−	4.17E+01±2.07E-14≈	4.17E+01±3.96E-14≈	4.01E+01±8.35E+00≈	4.17E+01±2.20E-14
C11	-1.52E-03±3.54E-18≈	-1.52E-03±4.36E-18≈	-1.52E-03±3.16E-18≈	-1.52E-03±8.14E-16≈	-1.52E-03±3.77E-18
C12	-1.99E-01±5.92E-17≈	-5.26E+00±2.29E+01+	-6.30E-01±2.15E+00≈	-1.99E-01±3.58E-17≈	-1.99E+00±4.81E-17
C13	-6.84E+01±2.90E-14≈	-6.84E+01±2.90E-14≈	-6.84E+01±2.90E-14≈	-6.84E+01±2.90E-14≈	-6.84E+01±2.90E-14
C14	1.59E-01±7.97E-01−	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C15	5.78E+13±5.10E+13−	1.91E+13±2.43E+13−	2.94E+00±1.50E+00≈	3.38E+00±1.02E+00−	2.94E+00±1.50E+00
C16	8.71E-01±1.79E-01−	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C17	3.73E-10±3.94E-10≈	2.27E-10±3.55E-10≈	6.93E-11±2.54E-10≈	4.40E-11±8.58E-11≈	2.05E-11±4.44E-11
C18	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
−	**8**	**2**	**1**	**4**	/
+	**1**	**2**	**0**	**0**	/
≈	**9**	**14**	**17**	**14**	/

TABLE S-11
EXPERIMENTAL RESULTS OF DECODE WITH FIVE VARYING FP OVER 25 INDEPENDENT RUNS ON THE 18 TEST FUNCTIONS WITH 30D FROM IEEE CEC2010

IEEE CEC2010 with 30D	$FP = 0.55$ Mean OFV±Std Dev	$FP = 0.65$ Mean OFV±Std Dev	$FP = 0.75$ Mean OFV±Std Dev	$FP = 0.95$ Mean OFV±Std Dev	$FP = 0.85$ Mean OFV±Std Dev
C01	-8.18E-01±6.29E-03≈	-8.19E-01±3.04E-03≈	-8.20E-01±2.01E-03≈	-8.13E-01±9.94E-03−	-8.19E-01±3.20E-03
C02	-2.24E+00±1.70E-02≈	-2.25E+00±1.82E-02≈	-2.25E+00±1.65E-02≈	-2.25E+00±1.61E-02≈	-2.23E+00±3.49E-02
C03	1.95E+01±1.37E+01≈	1.72E+01±1.43E+01≈	2.18E+01±1.25E+01≈	2.41E+01±1.07E+01≈	2.06E+01±1.31E+01
C04	-3.33E-06±9.84E-12≈	-3.33E-06±1.45E-11≈	-3.33E-06±2.18E-12≈	-3.33E-06±2.75E-12≈	-3.33E-06±6.92E-12
C05	-4.83E+02±1.37E-01≈	-4.83E+02±1.37E-01≈	-4.83E+02±1.43E-01≈	-4.83E+02±1.99E-01≈	-4.83E+02±1.53E-01
C06	-5.27E+02±1.60E+00≈	-5.27E+02±1.30E+00≈	-5.27E+02±1.97E+00≈	-5.27E+02±1.24E+00≈	-5.28E+02±1.46E+00
C07	1.59E-01±7.97E-01−	5.43E-28±2.72E-27≈	1.59E-01±7.97E-01−	1.30E-27±4.55E-27≈	0.00E+00±0.00E+00
C08	5.43E-28±2.72E-27≈	2.34E+01±1.17E+02−	7.34E+00±2.54E+01−	8.53E-29±4.27E-28≈	0.00E+00±0.00E+00
C09	1.38E+01±3.96E+01−	6.20E+00±1.95E+01≈	3.75E+00±1.48E+01≈	7.04E+00±1.65E+00≈	8.97E+00±2.32E+01
C10	3.13E+01±4.03E-05≈	3.13E+01±3.19E-05≈	3.13E+01±2.45E-05≈	3.01E+01±6.26E+00≈	3.13E+01±1.72E-05
C11	-3.92E-04±4.13E-08≈	-3.92E-04±4.03E-10≈	-3.92E-04±3.75E-10≈	-3.92E-04±2.40E-10≈	-3.92E-04±3.11E-10
C12	-1.99E-01±7.14E-09≈	-1.99E-01±1.79E-08≈	-1.99E-01±9.29E-08≈	-1.99E-01±3.66E-08≈	-1.99E-01±1.23E-06
C13	-6.78E+01±1.15E+00≈	-6.77E+01±1.02E+00≈	-6.74E+01±1.32E+00≈	-6.71E+01±1.50E+00≈	-6.73E+01±1.60E+00
C14	3.63E-27±1.60E-26≈	1.59E-01±7.97E-01−	1.27E-29±6.37E-29≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C15	2.16E+01±5.28E-07≈	2.16E+01±2.97E-07≈	2.16E+01±8.49E-07≈	2.18E+01±1.14E+00≈	2.18E+01±1.14E+00
C16	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C17	8.54E-02±1.52E-01≈	6.26E-02±1.38E-01≈	4.60E-02±9.46E-02≈	1.16E-01±1.59E-01−	4.48E-02±1.21E-01
C18	2.67E+00±1.33E+01−	1.52E+00±7.27E+00−	9.10E-04±4.55E-03−	1.89E-09±9.44E-09+	3.03E-06±1.29E-05
−	**3**	**3**	**3**	**2**	/
+	**0**	**0**	**0**	**1**	/
≈	**15**	**15**	**15**	**15**	/

TABLE S-12

EXPERIMENTAL RESULTS OF DECODE WITH SIX VARYING η_L OVER 25 INDEPENDENT RUNS ON THE 18 TEST FUNCTIONS WITH 30D FROM IEEE CEC2010. FR DENOTES THE FEASIBLE RATE.

IEEE CEC2010 with 30D	$\eta_L = 10^{-6}$ Mean OFV±Std Dev (FR)	$\eta_L = 10^{-10}$ Mean OFV±Std Dev (FR)	$\eta_L = 10^{-14}$ Mean OFV±Std Dev (FR)	$\eta_L = 10^{-22}$ Mean OFV±Std Dev (FR)	$\eta_L = 10^{-26}$ Mean OFV±Std Dev (FR)	$\eta_L = 10^{-18}$ Mean OFV±Std Dev (FR)
C01	-8.11E-01±9.73E-03−	-8.19E-01±4.09E-03≈	-8.19E-01±3.90E-03≈	-8.19E-01±4.12E-03≈	-8.18E-01±6.28E-03≈	-8.19E-01±3.20E-03
C02	-2.25E+00±2.40E-02≈	-2.25E+00±1.60E-02≈	-2.25E+00±2.17E-02≈	-2.25E+00±2.56E-02≈	-2.25E+00±1.47E-02≈	-2.23E+00±3.49E-02
C03	1.38E+01±1.46E+01≈	2.18E+01±1.25E+01≈	2.29E+01±1.17E+01≈	2.29E+01±1.17E+01≈	2.41E+01±1.07E+01≈	2.06E+01±1.31E+01
C04	-3.33E-06±1.32E-11≈	-3.33E-06±6.90E-12≈	-3.33E-06±4.38E-12≈	-3.33E-06±1.49E-11≈	-3.33E-06±1.34E-11≈	-3.33E-06±6.92E-12
C05	(96%)−	-4.83E+02 ±1.38E-01≈	-4.83E+02±1.44E-01≈	-4.83E+02±2.70E-01≈	-4.83E+02±1.30E-01≈	-4.83E+02±1.53E-01
C06	-5.27E+02±2.05E+00≈	-5.27E+02±1.92E+00≈	-5.27E+02±2.10E+00≈	-5.28E+02±1.15E+00≈	-5.28E+02±1.55E+00≈	-5.28E+02±1.46E+00
C07	5.43E-28±2.72E-27≈	1.59E-01±7.97E-01−	6.38E-01±1.49E+00−	3.19E-01±1.10E+00−	6.38E-01±2.30E+00−	0.00E+00±0.00E+00
C08	7.60E+00±2.63E+01−	2.15E+01±7.65E+01−	1.09E-27±3.76E-27≈	1.14E-27±3.95E-27≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C09	3.47E+00±1.47E+01≈	4.76E+00±8.80E+00≈	6.87E+00±1.92E+01≈	2.77E+00±1.39E+01≈	3.59E+00±1.44E+01≈	8.97E+00±2.32E+01
C10	3.13E+01±1.68E-05≈	3.13E+01±3.86E-05≈	3.13E+01±2.13E-05≈	3.13E+01±4.56E-05≈	3.13E+01±1.46E-05≈	3.13E+01±1.72E-05
C11	-3.92E-04±8.82E-08≈	-3.92E-04±5.74E-09≈	-3.92E-04±4.13E-10≈	-3.92E-04±2.29E-10≈	-3.92E-04±2.05E-10≈	-3.92E-04±3.11E-10
C12	-1.98E-01±7.87E-03≈	-1.99E+00±4.58E-06≈	-1.99E-01±6.12E-09≈	-1.97E-01±9.65E-03≈	-1.96E-01±1.33E-02≈	-1.99E-01±1.23E-06
C13	-6.76E+01±1.31E+00≈	-6.75E+01±1.41E+00≈	-6.79E+01±8.42E-01≈	-6.78E+01±1.16E+00≈	-6.77E+01±1.05E+00≈	-6.73E+01±1.60E+00
C14	1.59E-01±7.97E-01−	2.22E-28±1.11E-27≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	5.57E-29±2.78E-28≈	0.00E+00±0.00E+00
C15	2.16E+01±3.68E-07≈	2.16E+01±2.99E-07≈	2.16E+01±5.39E-07≈	2.16E+01±3.46E-07≈	2.16E+01±3.27E-07≈	2.18E+01±1.14E+00
C16	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C17	5.73E-02±1.22E-01≈	9.00E-02±1.57E-01≈	2.41E-01±5.95E-01−	8.58E-02±3.37E-01≈	7.80E-02±1.48E-01≈	4.48E-02±1.21E-01
C18	1.22E-14±5.98E-14+	3.83E-11±1.92E-10+	7.48E-12±3.74E-11+	6.01E-05±3.00E-04−	1.48E-14±5.31E-14+	3.03E-06±1.29E-05
−	**4**	**2**	**2**	**2**	**1**	/
+	**1**	**1**	**1**	**0**	**1**	/
≈	**13**	**15**	**15**	**16**	**16**	/

TABLE S-13

EXPERIMENTAL RESULTS OF DECODE WITH FIVE VARYING μ OVER 25 INDEPENDENT RUNS ON THE 18 TEST FUNCTIONS WITH 30D FROM IEEE CEC2010. FR DENOTES THE FEASIBLE RATE.

IEEE CEC2010 with 30D	$\mu = 10^{0}$ Mean OFV±Std Dev (FR)	$\mu = 10^{-3}$ Mean OFV±Std Dev (FR)	$\mu = 10^{-9}$ Mean OFV±Std Dev (FR)	$\mu = 10^{-12}$ Mean OFV±Std Dev (FR)	$\mu = 10^{-6}$ Mean OFV±Std Dev (FR)
C01	-8.17E-01±5.91E-03≈	-8.18E-01±4.87E-03≈	-8.17E-01±5.70E-03≈	-8.19E-01±3.64E-03≈	-8.19E-01±3.20E-03
C02	(0%)−	-2.24E+00±2.21E-02≈	-2.25E+00±.92E-02≈	-2.25E+00±2.05E-02≈	-2.23E+00±3.49E-02
C03	(0%)−	(0%)−	2.41E+01±1.07E+01≈	2.06E+01±1.31E+01≈	2.06E+01±1.31E+01
C04	(0%)−	(0%)−	-3.33E-06±3.13E-12≈	-3.33E-06±4.58E-12≈	-3.33E-06±6.92E-12
C05	(0%)−	(76%)−	-4.83E+02±1.91E-01≈	-4.83E+02±1.98E-01≈	-4.83E+02±1.53E-01
C06	(0%)−	-5.27E+02±1.43E+00≈	-5.28E+02±1.17E+00≈	-5.26E+02±1.99E+00≈	-5.28E+02±1.46E+00
C07	1.59E-01±7.97E-01−	5.43E-28±2.72E-27≈	1.59E-01±7.97E-01−	1.59E-01±7.97E-01−	0.00E+00±0.00E+00
C08	6.28E-28±2.73E-27≈	7.84E+00±2.72E+01−	1.59E-01±7.97E-01−	1.11E-27±3.83E-27≈	0.00E+00±0.00E+00
C09	(32%)−	9.36E+00±2.43E+01≈	3.24E+00±1.44E+01≈	3.09E+00±1.36E+01≈	8.97E+00±2.32E+01
C10	(40%)−	3.13E+01±2.12E-05≈	3.13E+01±2.21E-05≈	3.13E+01±2.52E-05≈	3.13E+01±1.72E-05
C11	(0%)−	(0%)−	-3.92E-04±.06E-10≈	-3.92E-04±1.21E-10≈	-3.92E-04±3.11E-10
C12	(0%)−	(0%)−	-1.99E-01±9.12E-09≈	-1.99E-01±2.00E-05≈	-1.99E-01±1.23E-06
C13	-6.80E+01±8.56E-01≈	-6.75E+01±1.15E+00≈	-6.76E+01±9.38E-01≈	-6.77E+01±9.25E-01≈	-6.73E+01±1.60E+00
C14	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	5.57E-29±2.78E-28≈	0.00E+00±0.00E+00
C15	2.16E+01±5.88E-07≈	2.16E+01±3.61E-07≈	2.18E+01±1.14E+00≈	2.16E+01±4.33E-07≈	2.18E+01±1.14E+00
C16	(92%)−	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00≈	0.00E+00±0.00E+00
C17	(52%)−	2.36E-01±6.44E-01−	3.48E+01±1.20E+02≈	(88%)−	4.48E-02±1.21E-01
C18	1.95E+0±1.24E+04−	1.41E-02±5.04E-02−	4.17E-06±1.97E-06≈	4.12E-11±2.06E-10+	3.03E-06±1.29E-05
−	**13**	**8**	**2**	**2**	/
+	**0**	**0**	**0**	**1**	/
≈	**5**	**10**	**16**	**15**	/